VIII. *On* ABEL'S *Theorem and Abelian Functions.*

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THE present paper is divided into two sections. The object of Section I. is to obtain an expression for an integral more general than, but intimately connected with, that occurring in ABEL'S theorem. The latter, as enunciated by Professor ROWE in his memoir in the Phil. Trans., 1881, is as follows:—If

$$\chi(x, y)=0$$

be a rational algebraical equation between x and y, then an expression can always be found for

$$\Sigma\int\frac{\mathrm{U}dx}{f(x)\frac{\delta\chi}{\delta y}}$$

where $f(x)$ is a function of x only, U a rational algebraical integral function of x and y, and the upper limits of the series of integrals are the roots of the eliminant with regard to y of $\chi(x, y)=0$ and a function $\theta(x, y)$.

In the case here considered two equations respectively of the degrees m and n between three variables

$$\mathrm{F}_m(x, y, z)=0$$
$$\mathrm{F}_n(x, y, z)=0$$

are given (these alone being considered, as subsequent generalisation to the case of r equations between r dependent variables and one independent variable is obvious); and an expression is obtained for

$$\Sigma\int\frac{\mathrm{U}dx}{f(x)\mathrm{J}\left(\frac{\mathrm{F}_m,\ \mathrm{F}_n}{y,\ z}\right)}$$

the upper limits of the integrals being given by the roots of the equation arrived at by eliminating y and z between F_m, F_n and an arbitrary equation

$$\mathrm{F}_p(x, y, z)=0$$

or, what is the same thing, by the co-ordinates x of the points of intersection of the three surfaces represented by F_m, F_n, F_p.

Some preliminary considerations (in connexion with §§ 92 *sqq.* of SALMON'S 'Higher Algebra') are adduced in reference to the eliminants of the three equations in each of the variables; thus if X be the equation in x obtained by eliminating y and z, it is expressed in the form

$$X=B_m F_m+B_n F_n+B_p F_p$$

which afterwards proves useful. Then the ordinary case (above referred to) of ABEL'S theorem is treated on the lines laid down in CLEBSCH and GORDAN'S 'Treatise on the Abelian Functions;' and under the guidance of this the more general form is investigated with the result

$$\Sigma\int\frac{\mathrm{U}dx}{f(x)\mathrm{J}\left(\frac{\mathrm{F}_m,\ \mathrm{F}_n}{y,\ z}\right)}=\Theta\left[\frac{1}{f(x)}\right]\Sigma\left\{\frac{\mathrm{U}}{\mathrm{J}}\log \mathrm{F}_p\right\}+\mathrm{C}$$

Θ being the symbol introduced by BOOLE.

The remainder of this section is occupied with the discussion of two examples of this theorem. In the first, expressions are obtained for $E(u_1+u_2+u_3)$ and $\Pi(u_1+u_2+u_3)$, E and Π being the second and third elliptic integrals; and in the second example $E(u_1+u_2+\ldots+u_7)$ is considered.

In Section II. the addition-theorem for the functions presented in WEIERSTRASS'S memoir ('Crelle,' t. lii., (1856), p. 285) is investigated. [It may be pointed out that the fundamental equations occur as natural examples of the more general form of ABEL'S theorem proved in Section I.; but the equations which are obtained almost immediately are identical with those used by WEIERSTRASS, and so this case does not belong distinctively to the form of ABEL'S theorem for the curve of double curvature.] For the purpose of the section use is made of the "integral-function," the partial differential coefficients of which with respect to the amplitudes give the squares of the Abelian functions. The theory is worked out at some length, and the necessary formulæ are deduced from the fundamental equations in a manner somewhat different from that of WEIERSTRASS. From the form first obtained for the sum of three integral-functions an important theorem is deduced in § 21, and a verification of this is afterwards furnished by the expansion of the two sides of the equation. It is then applied, as already mentioned, to obtain the addition-theorem for the functions.

In §§ 25, 26 is given the discussion of a particular case of the above, viz., when the functions are of the order 2, the fifteen functions being the quotients of all but one of the double theta-functions by that one. This has already formed the subject of a paper by CAYLEY in 'Crelle,' t. lxxxviii. (1878), p. 74.

SECTION I.

1. Before proceeding to the consideration of the theorem it is necessary to indicate the form in which the eliminant of three equations in three variables (or in general of μ equations in μ variables) will be used.

If we consider two equations in two variables say

$$\begin{aligned} f_n &\equiv x_0+x_1y+x_2y^2+\ldots+x_ny^n=0\\ \phi_m &\equiv X_0+X_1y+X_2y^2+\ldots+X_my^m=0 \end{aligned}$$

and if X be the eliminant of f_n and ϕ_m with regard to y, then we have

$$X=Af+B\phi.$$

Now X being of degree mn in x, A must be of degree $mn-n$ and B of $mn-m$; while it is sufficient that the highest power of y in A be the $(m-1)$th and in B the $(n-1)$th. Write then

$$\begin{aligned} A&=A_0+A_1y+A_2y^2+\ldots+A_{m-1}y^{m-1}\\ B&=B_0+B_1y+B_2y^2+\ldots+B_{n-1}y^{n-1}. \end{aligned}$$

Substitute in X; since X is explicitly free from y all the coefficients of powers of y in the result must be zero. This then gives

$$\begin{array}{llr} A_0x_1+A_1x_0 & +B_0X_1+B_1X_0 & =0\\ A_0x_2+A_1x_1+A_2x_0 & +B_0X_2+B_1X_1+B_2X_0 & =0\\ A_0x_3+A_1x_2+A_2x_1+A_3x_0 & +B_0X_3+B_1X_2+B_2X_1+B_3X_0 & =0\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \end{array}$$

$m+n-1$ equations to determine the ratios of the $m+n$ quantities A, B. Let

$$E=\left|\begin{array}{cccccccc} 1 & y & y^2\ldots & y^{m-1} & 0 & 0 & \ldots & 0\\ x_1 & x_0 & 0\ldots & 0 & X_1 & X_0 & \ldots & \\ x_2 & x_1 & x_0\ldots & 0 & X_2 & X_1 & X_0\ldots & \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \end{array}\right|$$

$$F=\left|\begin{array}{cccccccc} 0 & 0\ldots & & 0 & 1 & y & y^2\ldots & y^{n-1}\\ x_1 & x_0 & 0\ldots & & X_1 & X_0 & 0\ldots & \\ x_2 & x_1 & x_0\ldots & & X_2 & X_1 & X_0\ldots & \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \end{array}\right|$$

Then A_r bears to the minor of y^r in E the same ratio as B_s bears to the minor of y^s in F: thus

$$\frac{A}{E}=\frac{B}{F}.$$

But the diagonal term in E is

$$x_0^{m-1}X_m^{\ n}$$

and E is therefore of the degree $n(m-1)$, *i.e.*, of the same degree in X as A; it is obviously of the same degree in y; hence $\frac{\mathrm{A}}{\mathrm{E}}$ is merely an arithmetical constant, and we may write

$$\mathrm{A}=\mathrm{E}, \quad \mathrm{B}=\mathrm{F}.$$

2. When we come to apply this method to the formation of the eliminant with regard to y and z of three equations F_1, F_2, F_3 in three variables x, y, z the result, though of similar form, viz.:

$$\mathrm{A}_{x,1}\mathrm{F}_1+\mathrm{A}_{x,2}\mathrm{F}_2+\mathrm{A}_{x,3}\mathrm{F}_3,$$

can in general be obtained neither so directly nor without the help of the considerations in SALMON'S 'Higher Algebra,' §§ 92, *sqq.* If the three equations be each of the degree 2, the method will apply exactly as in the preceding paragraph and we obtain

$$\mathrm{X}=\mathrm{A}_{x,1}\mathrm{F}_1+\mathrm{A}_{x,2}\mathrm{F}_2+\mathrm{A}_{x,3}\mathrm{F}_3$$

where

$$\mathrm{A}_{x,r}=\alpha_{x,r}+\beta_{x,r}x+\gamma_{x,r}y\,;$$

but if the equations be not of this degree, then the following is our rule. Let F_1, F_2, F_3 be of the degrees m, n, p respectively: then we form all possible equations, which the variables satisfy, of degree not higher than $m+n+p-2$: thus we multiply F_1 by

$$y^{n+p-2},\ y^{n+p-3},\ \ldots,\ y^{n+p-3}z,\ y^{n+p-4}z,\ \ldots,\ y^{n+p-4}z^2,\ y^{n+p-5}z^2,\ \ldots$$

and so on; and so we obtain

$$\tfrac{1}{2}(n+p-1)(n+p)+\tfrac{1}{2}(p+m-1)(p+m)+\tfrac{1}{2}(m+n)(m+n-1)$$

equations from which to eliminate

$$\tfrac{1}{2}(m+n+p-1)(m+n+p)$$

quantities. But these equations are not all independent, being connected by a number of identities of the form

$$z^r y^s \mathrm{F}_1.\mathrm{F}_2=z^r y^s \mathrm{F}_2.\mathrm{F}_1$$

(where $r+s\leqq p-2$), of which there are $\frac{1}{2}p(p-1)$; there are $\frac{1}{2}n(n-1)$ of the form

$$z^{r'}y^{s'}\mathrm{F}_1.\mathrm{F}_3=z^{r'}y^{s'}\mathrm{F}_3.\mathrm{F}_1 \text{ where } r'+s'\leqq n-2$$

and $\frac{1}{2}m(m-1)$

$$z^{r''}y^{s''}\mathrm{F}_2.\mathrm{F}_3=z^{r''}y^{s''}\mathrm{F}_3.\mathrm{F}_2 \text{ where } r''+s''\leqq m-2,$$

and thus we have the proper number of equations. To find the eliminant X we write down the coefficients (which are, of course, functions of x) in the $\frac{1}{2}(n+p-1)(n+p)+\ldots$ equations; and reduce them to the form of a determinant by adding the coefficients in the $\frac{1}{2}p(p-1)+\ldots$ equations; then the eliminant is the quotient of the determinant formed by any

$$\tfrac{1}{2}(m+n+p-1)(m+n+p)$$

rows of the set first written down by the determinant formed from the second set after the elision of these rows.

3. To show how this can be brought into the desired form the easiest plan will be to consider an example. Let

$$\begin{aligned}
&F_1 \equiv Ay+Bz+C\\
&F_2 \equiv A'y^2+F'yz+B'z^2+E'y+D'z+C'\\
&F_3 \equiv A''y^3+D''y^2z+E''yz^2+B''z^3+F''y^2+G''yz+J''z^2+H''y+K''z+C''
\end{aligned}$$

where the coefficient of the highest powers of z and y are constants and those of other powers are functions of x such as make the order of the highest expression in the term of the same order as the equation; thus, for instance

$$\begin{aligned}
F''&=fx+f'\\
H''&=hx^2+h'x+h''
\end{aligned}$$

and so on. Then we have, since $m=1$, $n=2$, $p=3$

	y^4	y^3z	yz^2	yz^3	z^4	y^3	yz	yz^2	z^3	y^2	yz	z^2	y	z	1	$yF_2.F_1$	$zF_2.F_1$	$F_2.F_1$	$F_3.F_1$
y^3F_1	A	B				C										A′			A″
y^2zF_1		A	B				C									F′	A′		D″
yz^2F_1			A	B				C								B′	F′		E″
z^3F_1				A	B				C								B′		B″
y^2F_1						A	B			C						E′		A′	F″
yzF_1							A	B			C					D′	E′	F′	G″
z^2F_1								A	B			C					D′	B′	J″
yF_1										A	B		C			C′.		E′	H″
zF_1											A	B		C			C′	D′	K″
F_1													A	B	C			C′	C″
y^2F_2	A′	F′	B′			E′	D′			C′						A			
yzF_2		A′	F′	B′			E′	D′			C′					B	A		
z^2F_2			A′	F′	B′			E′	D′			C′					B		
yF_2						A′	F′	B′		E′	D′		C′			C		A	
zF_2							A′	F′	B′		E′	D′		C′			C	B	
F_2										A′	F′	B′	E′	D′	C′			C	
yF_3	A″	D″	E″	B″		F″	G″	J″		H″	K″		C″						A
zF_3		A″	D″	E″	B″		F″	G″	J″		H″	K″		C″					B
F_3						A″	D″	E″	B″	F″	G″	J″	H″	K″	C″				C

To find the eliminant we choose any 15 rows (leaving out say the y^2F_1, yzF_1, zF_2, yF_3) and form a determinant, and then divide by

$$\begin{vmatrix} E' & 0 & A' & F'' \\ D' & E' & F' & G'' \\ 0 & C & B & 0 \\ 0 & 0 & 0 & A \end{vmatrix} = u.$$

(As the object is to illustrate the general case and not merely to get the result in this particular case we have not selected those rows which leave the denominator in the simplest form.) In the determinant of 15^2 constituents multiply each column by the quantity which stands at the head of it, add the results horizontally along all the rows, and replace the constituents of the last column by these new constituents which are, in order,

$$y^3F_1,\ y^2zF_1,\ yz^2F_1,\ z^3F_1,\ z^2F_1,\ yF_1,\ zF_1,\ F_1,\ y^2F_2,\ yzF_2,\ z^2F_2,\ yF_2,\ F_2,\ zF_3,\ F_3,$$

so that if we expand we have the numerator of our eliminant in the form

$$A'_{x,1}F_1+A'_{x,2}F_2+A'_{x,3}F_3$$

where the A's are determinants differing from the initial determinants in the last column alone; $A'_{x,1}$ has for its constituents there the coefficients of F_1 so long as F_1 occurs in the later form and then zeros; $A'_{x,2}$ those of F_2 where it occurs and elsewhere zeros; $A'_{x,3}$ those of F_3 where it occurs and elsewhere zeros. Moreover, we know that our eliminant is an integral function of x not extending in an infinite series; hence each of the coefficients A' must be divisible by u. If not, one of the F's (say F_1) must be so divisible; since u is a function of x only it follows that, when $u=0$, $F_1=0$ whatever z and y may be. We shall assume that such factors are removed before the investigation begins as they are useless for the purposes for which the functions are required; and hence we obtain our eliminant in the form

$$X=A_{x,1}F_1+A_{x,2}F_2+A_{x,3}F_3.$$

Similar remarks of course apply to Y and Z, the eliminants with regard to z and x, y and x.

4. We may also obtain the result as follows:

Between $F_m=0$, $F_p=0$ eliminate z and denote the eliminant by X_y; then, as we have already seen, X_y can be expressed in the form

$$X_y=\lambda_m F_m+\lambda_p F_p.$$

Between $F_n=0$, $F_p=0$ eliminate z and denote this eliminant by X'_y; then

$$X'_y=\mu_n F_n+\mu_p F_p.$$

Now X_y, X'_y are both functions of x and y; eliminating y between them and denoting the eliminant by X we have

$$\begin{aligned} X &= \rho_y X_y + \rho'_y X'_y \\ &= A_m F_m + A_n F_n + A_p F_p \end{aligned}$$

of the same form as before.

This method of expressing an eliminant obviously admits of generalisation to the case of r equations in r variables.

5. The preceding method enables us to obtain the eliminants of the equations with regard to the different variables in a particular form which is useful in the proof of the general theorem in § 7; but when the object is merely to obtain the equation giving the roots x_μ which are to form the upper limits of our integrals we should arrive at the result more easily as follows.

Obviously

$$X = \prod_{\mu=1}^{\mu=mn} F_p(x, y_\mu, z_\mu)$$

where y_μ, z_μ constitute one of the mn pairs of roots of the equations

$$F_m = 0 \quad F_n = 0$$

regarded as giving y, z in terms of x; and the product is taken over all these pairs. Now the coefficients on the right-hand side will be symmetric functions of y and z, and these can be evaluated (by the method given in SALMON's 'Higher Algebra,' § 74) in terms of x; and there will be obtained the required equation in x.

ABEL'S *Theorem.*

6. Let

$$\chi(x, y) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

be an equation of the degree n which gives y in terms of x; and let

$$\theta(x, y)$$

denote a function of x of degree m—reducible to degree $n-1$ at most in y by means of (i)—the coefficients of y in which are functions of x and contain any number of arbitrary constants. Treating $\chi=0$, $\theta=0$ as two equations to determine the values of the variables, these arbitrary constants will enter into the expressions for the values of x, and will therefore vary when the latter vary. Let such a variation take place, so that

$$\frac{\delta\theta}{\delta x}\,dx + \frac{\delta\theta}{\delta y}\,dy + \delta\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

δ operating only on the constants in θ. Moreover we have from (i)

$$\frac{\partial\chi}{\partial x}dx+\frac{\partial\chi}{\partial y}dy=0$$

or writing

$$dw=\frac{dx}{\dfrac{\partial\chi}{\partial y}}=\frac{dy}{-\dfrac{\partial\chi}{\partial x}}$$

the equation (ii) becomes

$$-dw\left(\frac{\partial\chi}{\partial x}\frac{\partial\theta}{\partial y}-\frac{\partial\theta}{\partial x}\frac{\partial\chi}{\partial y}\right)+\delta\theta=0$$

or

$$dw=\frac{\delta\theta}{J(\chi,\theta)}$$

and therefore

$$\Sigma U dw=\Sigma\frac{U}{J(\chi,\theta)}\delta\theta\,.\qquad\ldots\ldots\ldots\ldots\qquad\text{(iii)}$$

where U is any rational function of x and y, and the summation is taken over all the roots x_μ of the equation obtained by the elimination of y between χ and θ.

Let X, Y respectively denote the eliminants of χ, θ with regard to y, x; then we can express X, Y in the form

$$\left.\begin{aligned}X&=A\chi+B\theta\\Y&=C\chi+D\theta\end{aligned}\right\}\qquad\ldots\ldots\ldots\ldots\qquad\text{(iv)}$$

and we write

$$\Delta=AD-BC.$$

Now whatever the function U may be it can be written in the form

$$\frac{T}{f(x)};$$

for it must be expressible as

$$\frac{f_2(x,y_1)}{f_1(x,y_1)}$$

that is,

$$\frac{f_2(x,y_1)\prod\limits_{\mu=2}^{\mu=n}f_1(x,y_\mu)}{\prod\limits_{\mu=1}^{\mu=n}f_1(x,y_\mu)}$$

which by means of the equation $\chi=0$ is at once reduced to the above form; thus

$$\Sigma\frac{T}{f(x)}\frac{dx}{\dfrac{\partial\chi}{\partial y}}=\Sigma\frac{T}{f(x)}\frac{\delta\theta}{J(\chi,\theta)}\qquad\ldots\ldots\ldots\ldots\qquad\text{(v).}$$

Each of the equations $X=0$, $Y=0$ has mn roots; if $x=x_\mu$, $y=y_{\mu'}$ be a pair which make χ and θ both vanish, these may be called congruous; but if $x=x_\mu$, $y=y_{\mu'}$ do not make χ and θ both vanish then for these

$$\Delta=0,$$

while for congruous values Δ does not vanish.

Moreover, for congruous values of x and y we at once have

$$\frac{dX}{dx}\cdot\frac{dY}{dy}=\Delta . J(\chi, \theta)$$

the congruous values being substituted for the variables, so that

$$\frac{1}{J(\chi, \theta)}=\frac{\Delta}{\frac{dX}{dx}\cdot\frac{dY}{dy}}.$$

Now in (v) the summation is for all the x's and for one of the y's, say $y_{1,\mu}$, which may be regarded in the following way. When the equation

$$\chi(x, y)=0$$

is solved for y in terms of x, there will be n roots; take one of these and denote it by y_1, which is therefore a function of x. Substitute in turn $x_1, x_2, \ldots, x_{mn}$; then we obtain for y_1 a series of values, but all derived from the single root of χ. Thus

$$\Sigma\frac{T}{x-\alpha}\frac{\delta\theta}{J(\chi, \theta)}=\Sigma\frac{T_\mu}{x_\mu-\alpha}\,\delta\theta_\mu\frac{\Delta_\mu}{\frac{dX}{dx}\cdot\frac{dY}{dy}}$$

since we have $x=x_\mu$, $y=y_{1,\mu}$ $(\mu=1, 2, \ldots, mn)$ as the mn congruous roots. Moreover for roots other than these

$$\Delta=0$$

so that we may add on a number of vanishing terms to the right-hand side, and the removal of the restriction now gives

$$\Sigma\frac{T_\mu}{x_\mu-\alpha}\,\delta\theta_\mu\frac{\Delta_\mu}{\frac{dX}{dx_\mu}\,\frac{dY}{dy_{\mu'}}}$$

(where $\mu'=1,\mu$ or $2,\mu$ or . . . or n,μ).

Moreover from (iv)

$$\Delta\chi=DX-BY$$

and therefore

$$\Delta_\mu\frac{\partial\chi}{\partial y_\mu}=-B_{\mu,\mu'}\frac{dY}{dy_{\mu'}}$$

the term corresponding to a differential of X disappearing and the others vanishing in virtue of the values assigned to the two variables; thus our expression becomes

$$-\Sigma \frac{\mathrm{T}_\mu}{x_\mu-\alpha}\,\delta\theta_\mu \frac{\mathrm{B}_{\mu\mu'}}{\dfrac{\delta\chi}{\delta y_{\mu'}}\cdot\dfrac{d\mathrm{X}}{dx_\mu}}.$$

But the coefficient of $\frac{1}{x}$ in the expansion of $\frac{\mathrm{V}}{\mathrm{X}}$ in descending powers of x is

$$\begin{aligned}\mathrm{C}_{\frac{1}{x}}\,\Sigma \frac{\mathrm{V}_\iota}{(x-x_\iota)\dfrac{d\mathrm{X}}{dx_\iota}} \\ =\Sigma \frac{\mathrm{V}_\iota}{\dfrac{d\mathrm{X}}{dx_\iota}}\end{aligned}$$

and therefore the foregoing

$$=-\mathrm{C}_{\frac{1}{x}}\,\Sigma \frac{\mathrm{T}}{x-\alpha}\,\delta\theta \frac{\mathrm{B}_{\mu'}}{\mathrm{X}\dfrac{\delta\chi}{\delta y_{\mu'}}}$$

the Σ referring to the n values of y obtained from the equation (i), and the expansion being in the factors of X alone. But since we are substituting for y from (i) we have χ always zero in this, and therefore

$$\mathrm{X}=\mathrm{B}_{\mu'}\theta.$$

Taking now into account the expansion for the factor $\frac{1}{x-\alpha}$, we have finally

$$\Sigma \frac{\mathrm{T}}{x-\alpha}\,\frac{\delta\theta}{\mathrm{J}(\chi,\,\theta)}=-\mathrm{C}_{\frac{1}{x}}\,\Sigma \frac{\mathrm{T}}{x-\alpha}\,\frac{\delta\theta}{\theta}\,\frac{1}{\dfrac{\delta\chi}{\delta y}}+\Sigma\left[\frac{\mathrm{T}}{\dfrac{\delta\chi}{\delta y}}\,\frac{\delta\theta}{\theta}\right]_{x=\alpha}$$

the summation in each of the terms on the right-hand side being for the n values of y. Now on the introduction of BOOLE's symbol Θ (*cf.* Phil. Trans., 1857, p. 751), the right-hand side is merely the definition of

$$\Theta\left[\frac{1}{x-\alpha}\right]\Sigma\frac{\mathrm{T}}{\dfrac{\delta\chi}{\delta y}}\,\frac{\delta\theta}{\theta}.$$

Let $x=\alpha$ be a root of $f(x)=0$, and in (v) expand $[f(x)]^{-1}$ in a series of partial fractions corresponding to the roots; then expressions of the form $\frac{\mathrm{T}}{x-\alpha}\,\frac{\delta\theta}{\mathrm{J}(\chi,\,\theta)}$ are obtained. Moreover, from the nature of the preceding fractions and the definition of the symbol Θ in connexion with them, it is obviously a distributive symbol; thus we have

$$\Sigma\frac{T}{f(x)}\frac{dx}{\frac{\partial\chi}{\partial y}}=\Sigma\frac{T}{f(x)}\frac{\delta\theta}{J(\chi,\theta)}$$

$$=\Theta\left[\frac{1}{f(x)}\right]\Sigma\frac{T}{\frac{\partial\chi}{\partial y}}\frac{\delta\theta}{\theta}$$

where the summation on the left-hand side is over the roots of the equation

$$X=0$$

while on the right-hand side it is over the n roots y of the equation $\chi(x, y)=0$. The variables on the right-hand side being the arbitrary constants in θ which occur only in the factor $\frac{\delta\theta}{\theta}$, we may integrate, and we have as the result

$$\sum_{\mu=1}^{\mu=mn}\int^{x_\mu}\frac{T}{f(x)}\frac{dx}{\frac{\partial\chi}{\partial y}}=\Theta\left[\frac{1}{f(x)}\right]\sum_{y=y_1}^{y=y_n}\left\{\frac{T}{\frac{\partial\chi}{\partial y}}\log\theta\right\}+C$$

agreeing with the form given in Professor ROWE's memoir (Phil. Trans. 1881, p. 721).

7. In the generalisation of the theorem we shall consider only two dependent variables y, z and one independent variable x; it will be seen that the work would apply, *mutatis mutandis*, to $k-1$ dependent variables and one independent. Let the variables y and z be given as functions of x by the equations

$$F_m\,(x, y, z)=0$$
$$F_n\,(x, y, z)=0$$

of the degrees m and n respectively. Let

$$F_p\,(x, y, z)$$

be a function of x, y and z the coefficients of y and z in which are functions of x with any number of arbitrary constants; so that as in the simple case when z, x and y vary the constants also vary. Using the same notation as before we have

$$\frac{\partial F_m}{\partial x}dx+\frac{\partial F_m}{\partial y}dy+\frac{\partial F_m}{\partial z}dz=0,$$

$$\frac{\partial F_n}{\partial x}dx+\frac{\partial F_n}{\partial y}dy+\frac{\partial F_n}{\partial z}dz=0.$$

Therefore

$$\frac{dz}{\frac{\partial(F_m, F_n)}{\partial(x, y)}}=\frac{dx}{\frac{\partial(F_m, F_n)}{\partial(y, z)}}=\frac{dy}{\frac{\partial(F_m, F_n)}{\partial(z, x)}}=dw$$

and

$$\frac{\delta F_p}{\delta z}dz+\frac{\delta F_p}{\delta x}dx+\frac{\delta F_p}{\delta y}dy+\delta F_p=0$$

so that

$$dw.J\left(\frac{F_m,\ F_n,\ F_p}{x,\ y,\ z}\right)+\delta F_p=0$$

and therefore

$$-\Sigma T dw=\Sigma\frac{T}{J\left(\frac{F_m,\ F_n,\ F_p}{x,\ y,\ z}\right)}\delta F_p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i).$$

Let Z, X and Y be the eliminants of F_m, F_n and F_p respectively with regard to x and y, y and z, z and x; then as explained in § 3 we may write

$$\begin{aligned}X&=A_mF_m+A_nF_n+A_pF_p\\ Y&=B_mF_m+B_nF_n+B_pF_p\\ Z&=C_mF_m+C_nF_n+C_pF_p.\end{aligned}$$

Each of the equations $z=0$, $x=0$, $y=0$, has mnp solutions; let those values (mnp in all) which make F_m, F_n and F_p vanish simultaneously be called congruous.

Write

$$\Delta=\begin{vmatrix}A_m & A_n & A_p\\ B_m & B_n & B_p\\ C_m & C_n & C_p\end{vmatrix}$$

so that for non-congruous values Δ is zero.

Now whatever be the value of T it can be put into the form

$$\frac{\Psi(x,\ y,\ z)}{\Phi(x,\ y,\ z)}$$

where Φ and Ψ are rational integral algebraical functions of x, y and z; and this can be expressed as

$$\frac{U}{f(x)}$$

where U is an integral function of x, y and z and $f(x)$ is a function of x only. For it is

$$\frac{\Psi(x,\ y_1,\ z_1)\prod_{\mu=2}^{\mu=mn}\Phi(x,\ y_\mu,\ z_\mu)}{\prod_{\mu=1}^{\mu=mn}\Phi(x,\ y_\mu,\ z_\mu)}$$

where y_μ, z_μ are a pair of values of y and z which satisfy the equations

$$F_m=0,\quad F_n=0.$$

Now the denominator will consist of symmetric functions of the y's and z's, the coefficients in its different terms involving x. These can be expressed in terms of x alone,[*] while the new term in the numerator can be expressed in terms of symmetric functions of the y's and z's and of y_1 and z_1, and thus T is reduced to the form

$$\frac{\mathrm{U}}{f(x)}$$

and therefore

$$-\Sigma \mathrm{T}\,dw=\Sigma\frac{\mathrm{U}}{f(x)}\frac{\delta \mathrm{F}_p}{\mathrm{J}\left(\dfrac{\mathrm{F}_m,\ \mathrm{F}_n,\ \mathrm{F}_p}{x,\ y,\ z}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii).}$$

Let $x=\alpha$ be any root of $f(x)=0$; then, as before, we consider

$$-\Sigma\frac{\mathrm{U}}{x-\alpha}dw,$$

that is,

$$\Sigma\frac{\mathrm{U}}{x-\alpha}\frac{\delta \mathrm{F}_p}{\mathrm{J}},$$

the summation being for the mnp values of x; and a definite value of y and one of z are to be substituted in terms of x before the summation is effected.

Having these definite values of y and z (obtained from $\mathrm{F}_m=0$, $\mathrm{F}_n=0$) if in them we substitute in turn the mnp values of x, we shall have mnp congruous values and therefore all the congruous values. For these, as we easily see,

$$\frac{1}{\mathrm{J}\left(\dfrac{\mathrm{F}_m,\ \mathrm{F}_n,\ \mathrm{F}_n}{x,\ y,\ z}\right)}=\frac{\Delta}{\dfrac{d\mathrm{Z}}{dz}\,\dfrac{d\mathrm{X}}{dx}\,\dfrac{d\mathrm{Y}}{dy}}$$

and therefore

$$-\Sigma\frac{\mathrm{U}}{x-\alpha}dw=\Sigma\frac{\mathrm{U}}{x-\alpha}\,\delta \mathrm{F}_p\frac{\Delta}{\dfrac{d\mathrm{Z}}{dz}\,\dfrac{d\mathrm{X}}{dx}\,\dfrac{d\mathrm{Y}}{dy}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

the summation on each side being the same. But for all values not included in this summation we have $\Delta=0$, and therefore the restrictions on the right-hand side may be removed without altering its value, and we shall consider the summation to extend over all the roots of $\mathrm{F}_m=0$, $\mathrm{F}_n=0$ considered as equations in y and z and over all the roots x.

Let α denote the minor of A, β that of B, γ that of C (in each case with the same suffix) in Δ. Then we have

$$\alpha_m\mathrm{X}+\beta_m\mathrm{Y}+\gamma_m\mathrm{Z}\equiv\Delta \mathrm{F}_m$$
$$\alpha_n\mathrm{X}+\beta_n\mathrm{Y}+\gamma_n\mathrm{Z}\equiv\Delta \mathrm{F}_n$$
$$\alpha_p\mathrm{X}+\beta_p\mathrm{Y}+\gamma_p\mathrm{Z}\equiv\Delta \mathrm{F}_p.$$

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[*] *Cf.* SALMON'S 'Higher Algebra,' § 74.

Differentiating the first two of these with respect to y and z separately and then inserting the values of x, y and z as they now occur on the right-hand side of (iii) we have

$$\beta_m\frac{dY}{dy}=\Delta\frac{\delta F_m}{\delta y}$$

$$\gamma_m\frac{dZ}{dz}=\Delta\frac{\delta F_m}{\delta z}$$

$$\beta_n\frac{dY}{dy}=\Delta\frac{\delta F_n}{\delta y}$$

$$\gamma_n\frac{dZ}{dz}=\Delta\frac{\delta F_n}{\delta z}$$

and therefore

$$\begin{aligned}\Delta^2\frac{\delta(F_m, F_n)}{\delta(y, z)}&=(\beta_m\gamma_n-\beta_n\gamma_m)\frac{dY}{dy}\frac{dZ}{dz}\\&=A_p\Delta\frac{dY}{dy}\frac{dZ}{dz}\end{aligned}$$

by a known theorem in determinants; thus (iii) becomes

$$-\Sigma\frac{U}{x-\alpha}\delta w=\Sigma\frac{U}{x-\alpha}\delta F_p\frac{A_p}{\frac{dX}{dx}}\frac{1}{\frac{\delta(F_m, F_n)}{\delta(y, z)}}.$$

Now expanding in partial fractions we have

$$\frac{K}{X}=\Sigma\frac{K_\iota}{(x-x_\iota)\frac{dX}{dx_\iota}}$$

and therefore the right-hand side becomes

$$C_1\sum_x\left[\frac{U}{x-\alpha}\frac{\delta F_p}{\frac{\delta(F_m, F_n)}{\delta(y, z)}}\frac{A_p}{X}\right]$$

considered as expanded for the factors of X alone or, including in the expansion the term arising from $\frac{1}{x-\alpha}$, it is equal to

$$C_1\sum_x\left[\frac{U}{x-\alpha}\frac{\delta F_p}{\frac{\delta(F_m, F_n)}{\delta(y, z)}}\frac{A_p}{X}\right]-\Sigma\left[\frac{U\delta F_p}{\frac{\delta(F_m, F_n)}{\delta(y, z)}}\frac{A_p}{X}\right]_{x=\alpha}$$

wherein the Σ implies summation for all values of y and z in terms of x derived from the equations $F_m=0$ and $F_n=0$. Since these values are to be substituted we have

$$\begin{aligned}X&=A_mF_m+A_nF_n+A_pF_p\\&=A_pF_p\end{aligned}$$

and therefore substituting this in the above which will replace the right-hand side of (iii) and inserting the value of dw the equation becomes

$$\Sigma\frac{U}{x-\alpha}\frac{dx}{J\left(\frac{F_m,\ F_n}{y,\ z}\right)}=\Sigma\left[\frac{U}{J\left(\frac{F_m,\ F_n}{y,\ z}\right)}\frac{\delta F_p}{F_p}\right]_{x=\alpha}-C_{\frac{1}{x}}\Sigma\left[\frac{U}{x-\alpha}\frac{1}{J\left(\frac{F_m,\ F_n}{y,\ z}\right)}\frac{\delta F_p}{F_p}\right]$$

$$=\Theta\left[\frac{1}{x-\alpha}\right].\Sigma\left[\frac{U}{J\left(\frac{F_m,\ F_n}{y,\ z}\right)}\frac{\delta F_p}{F_p}\right]$$

by the use of BOOLE's symbol Θ as before. The summation on the left-hand side is of course over the mnp roots x; on the right-hand side it is over the mn roots y and z in terms of x of the equations $F_m=0$ and $F_n=0$. We may obviously integrate as before; and using the distributive property of Θ we obtain as our result

$$\sum_{\mu=1}^{\mu=mnp}\int^{x_\mu}\frac{U}{f(x)}\frac{dx}{J\left(\frac{F_m,\ F_n}{y,\ z}\right)}-C=\Theta\left[\frac{1}{f(x)}\right].\sum_{\mu=1}^{\mu=mn}\left\{\frac{U}{J\left(\frac{F_m,\ F_n}{y,\ z}\right)}\log F_p\right\}\quad .\ \ .\ \ .\quad \text{(iv)}.$$

8. The general theorem will proceed on lines not widely different from the above, and may be enunciated as follows. Let

$$\begin{aligned}&F_1\ \ (x_1,\ x_2,\ .\ .\ .\ ,\ x_r)=0\\&F_2\ \ (x_1,\ x_2,\ .\ .\ .\ ,\ x_r)=0\\&\qquad .\ .\ .\ .\ .\ .\\&F_{r-1}(x_1,\ x_2,\ .\ .\ .\ ,\ x_r)=0\end{aligned}$$

be $r-1$ equations, of degrees $m_1,\ m_2,\ .\ .\ .\ ,\ m_{r-1}$ respectively, giving $x_2,\ .\ .\ .\ ,\ x_r$ in terms of x_1; and let

$$F_r(x_2,\ x_3,\ .\ .\ .\ ,\ x_r)$$

be a function of these dependent variables, the coefficients of which are functions of x_1 containing any number of arbitrary constants. Form the eliminant E of all the F's so that we shall obtain the set of roots x_1 by equating E to zero; and denote by U any algebraical rational integral function of $x_1,\ x_2,\ .\ .\ .\ x_r$. Then

$$\Sigma\int\frac{U}{f(x_1)}\frac{dx_1}{J\left(\frac{F_1,\ F_2,\ .\ .\ .\ ,\ F_{r-1}}{x_2,\ x_3,\ .\ .\ .\ ,\ x_r}\right)}=\Theta\left[\frac{1}{f(x_1)}\right].\Sigma\left\{\frac{U\log F_r}{J\left(\frac{F_1,\ F_2,\ .\ .\ .\ ,\ F_{r-1}}{x_2,\ x_3,\ .\ .\ .\ ,\ x_r}\right)}\right\}+A$$

the summation on the left-hand side being taken over all the roots of the equation $E=0$, which are assumed as the upper limits of the integrals, while on the right-hand side the summation is over all the roots of $F_1=0$, $F_2=0$, . . . , $F_{r-1}=0$ considered as $r-1$ simultaneous equations giving x_2. x_3, . . . , x_r in terms of x_1.

9. Returning now to (iv) the conditions that the right-hand side should reduce to a constant are :—

1°: That $f(x)\equiv 1$, or be a factor of U ;

and 2°:

$$\underset{x}{C_1}\,\Sigma\frac{U}{J}\log\theta=0$$

which will be satisfied by

$$\underset{x}{C_1}\,\frac{U}{J}=0.$$

Now J is of the order $m+n-2$ in x, and therefore the order of U may not be greater than $m+n-4$. In this case the number of terms in it will be

$$\frac{m+n-1.m+n-2.m+n-3}{1.2.3}\,.$$

But if the integrals $\int\frac{x^k}{J}dx$ be formed they are not all independent for

$$\int\frac{V_nF_m}{J}dx=0$$

$$\int\frac{V_mF_n}{J}dx=0$$

where V_m and V_n are arbitrary functions of the orders $m-4$ and $n-4$ respectively, and contain

$$\frac{m-1.m-2.m-3}{1.2.3}\text{ and }\frac{n-1.n-2.n-3}{1.2.3}$$

terms. Hence the number of independent integrals in the case when the right-hand side reduces to a constant or to zero is

$$\frac{m+n-1.m+n-2.m+n-3}{1.2.3}-\frac{m-1.m-2.m-3}{1.2\,3}-\frac{n-1.n-2.n-3}{1.2.3}=\tfrac{1}{2}mn(m+n-4)+1$$

This assumes that the surfaces F_m and F_n are the most general of the degrees m and n respectively and so possess no special singularities.

10. ABEL's theorem in the more simple case applies to the intersection of plane curves. There is a fixed curve given by

$$\chi(x,\,y)=0$$

and integrals connected with this curve are taken, the upper limits assigned being the abscissæ of its points of intersection with another curve, the equation to which

$$\theta(x, y)=0$$

contains a number of variable parameters and therefore represents a variable curve.

But the more general form of the theorem extends the application to curves in space. We take the curve which is the intersection of two surfaces

$$\mathrm{F}_m(x, y, z)=0$$
$$\mathrm{F}_n(x, y, z)=0$$

(and which will, as a rule, be a tortuous curve), and forming the corresponding integrals we assign as the upper limits of these the ordinates x of the points of intersection of this tortuous curve with a surface the equation to which

$$\mathrm{F}_p(x, y, z)=0$$

containing a number of variable parameters represents a variable surface.

The discussion of this geometrical interpretation and of the deductions to which it leads has been carried out in a memoir by CLEBSCH ('Crelle,' t. lxiii., p. 189, 1863), wherein he proceeds from the theorems which are the forms of (iv) and (v) when the right-hand sides are zero. Example I. which follows was suggested by an analogous geometrical illustration which Professor CAYLEY gave in one of his lectures at Cambridge in the Michaelmas Term, 1881, wherein he pointed out how to obtain $\mathrm{sn}(u+v+w)$ from the analytical expression for the co-planarity of the four points of intersection of an arbitrary plane (corresponding to $\mathrm{F}_p=0$) with a fixed tortuous curve in space which was the intersection of a circular cylinder and an elliptic cylinder respectively corresponding to $\mathrm{F}_m=0$ and $\mathrm{F}_n=0$.

We now proceed to consider two examples of (iv).

11. *Example I.*

Let

$$\mathrm{F}_m\equiv y^2-(1-x^2)=0$$
$$\mathrm{F}_n\equiv z^2-(1-k^2x^2)=0$$
$$\mathrm{F}_p\equiv \mathrm{A}x+\mathrm{B}y+\mathrm{C}z-1.$$

The eliminant X is obviously

$$\mathrm{X}=\Pi\{\mathrm{A}x-1\pm\mathrm{B}(1-x^2)^{\frac{1}{2}}\pm\mathrm{C}(1-k^2x^2)^{\frac{1}{2}}\}$$

Π denoting the product of the four expressions which the above includes owing to the two double signs. It is evidently of the fourth degree in x; let the roots be x_1, x_2, x_3, x_4. As there are three arbitrary constants there will be one relation between these four roots, and this can be exhibited in the form

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}=0.$$

If we choose $A=0$ and one of the two, B and C, to be unity and the other zero, $X=0$ reduces to

$$x^4=0$$

and we may therefore take zero as the lower limits of all our integrals.

Let

$$y_1=+(1-x^2)^{\frac{1}{2}}$$
$$z_1=+(1-k^2x^2)^{\frac{1}{2}}$$

then $\pm y_1$ are the roots of F_m, $\pm z_1$ those of F_n. We have

$$J=4y_1z_1=4(1-x^2.1-k^2x^2)^{\frac{1}{2}}$$

and therefore by our formula (iv)

$$\sum_{\mu=1}^{\mu=4}\int_0^{x_\mu}\frac{U\,dx}{f(x)\sqrt{(1-x^2)(1-k^2x^2)}}=\Theta\left[\frac{1}{f(x)}\right]\Sigma\left\{\frac{U}{yz}\log\left(1-Ax-By-Cz\right)\right\}$$

in which the Σ on the right-hand side implies summation for the expressions obtained by the substitutions

$$\begin{aligned} y&=y_1 \text{ and } z=z_1, \\ y&=-y_1 \quad ,, \quad z=z_1, \\ y&=y_1 \quad ,, \quad z=-z_1, \\ y&=-y_1 \quad ,, \quad z=-z_1. \end{aligned}$$

(i.) Let $f(x)\equiv 1$, $U=1$; then the right-hand side vanishes and we have

$$u_1+u_2+u_3+u_4=0$$

where

$$x=\operatorname{sn}u.$$

Thus the preceding determinantal relation will give $\operatorname{sn}(u_1+u_2+u_3)$, which is $-x_4$, in terms of the elliptic functions of u_1, u_2, u_3.

(ii.) Let $f(x)\equiv 1$, $U=z^2=1-k^2x^2$; then we have

$$\begin{aligned} &E(u_1)+E(u_2)+E(u_3)+E(u_4) \\ &=-C_{\frac{1}{x}}\,\Sigma\left\{\frac{z}{y}\log\left(1-Ax-By-Cz\right)\right\} \\ &=-C_{\frac{1}{x}}\left[\frac{z_1}{y_1}\log\left(\frac{1-Ax-Cz_1-By_1}{1-Ax-Cz_1+By_1}\right)+\frac{z_1}{y_1}\log\left(\frac{1-Ax+Cz_1+By_1}{1-Ax+Cz_1-By_1}\right)\right]. \end{aligned}$$

Expanding the logarithms on the right-hand side the n^{th} term gives

$$\frac{2}{2n-1}\,\mathrm{C}_{\frac{1}{x}}\left[z_1\,\frac{\mathrm{B}^{2n-1}y_1{}^{2n-2}}{(1-\mathrm{A}x-\mathrm{C}z_1)^{2n-1}}-z_1\,\frac{\mathrm{B}^{2n-1}y_1{}^{2n-2}}{(1-\mathrm{A}x+\mathrm{C}z_1)^{2n-1}}\right]$$

$$=\frac{2\mathrm{B}^{2n-1}}{2n-1}\,\mathrm{C}_{\frac{1}{x}}\left[y_1{}^{2n-2}\,\frac{z_1(1-\mathrm{A}x+\mathrm{C}z_1)^{2n-1}-z_1(1-\mathrm{A}x-\mathrm{C}z_1)^{2n-1}}{\{(1-\mathrm{A}x)^2-\mathrm{C}^2z_1{}^2\}^{2n-1}}\right]$$

$$=\frac{4\mathrm{B}^{2n-1}\mathrm{C}}{2n-1}\,\mathrm{C}_{\frac{1}{x}}\left[(1-x^2)^{n-1}(1-k^2x^2)\,\frac{\left\{(2n-1)(1-\mathrm{A}x)^{2n-2}+\dfrac{2n-1.2n-2.2n-3}{1.2.3}(1-\mathrm{A}x)^{2n-4}\mathrm{C}^2(1-k^2x^2)+\ldots\right\}}{(\mathrm{A}^2+k^2\mathrm{C}^2)^{2n-1}x^{4n-2}\left\{1-\dfrac{2\mathrm{A}}{\mathrm{A}^2+k^2\mathrm{C}^2}\,\dfrac{1}{x}+\ldots\right\}^{2n-1}}\right]$$

$$=\frac{(-1)^n4k^2\mathrm{B}^{2n-1}\mathrm{C}}{(2n-1)(\mathrm{A}^2+k^2\mathrm{C}^2)^{2n-1}}\,\mathrm{C}_{\frac{1}{x}}\left[\left\{1+\frac{2(2n-1)\mathrm{A}}{\mathrm{A}^2+k^2\mathrm{C}^2}.\frac{1}{x}-\ldots\right\}\right.$$
$$\left\{(2n-1)\mathrm{A}^{2n-2}-(2n-1)(2n-2)\mathrm{A}^{2n-3}\frac{1}{x}-k^2\mathrm{C}^2\left(\frac{2n-1.2n-2.2n-3}{1.2.3}\mathrm{A}^{2n-4}\right.\right.$$
$$\left.\left.\left.-\frac{2n-1\ldots 2n-4}{1.2.3}\mathrm{A}^{2n-5}\frac{1}{x}\right)+\ldots\right\}\right]$$

$$=\frac{(-1)^n4k^2\mathrm{B}^{2n-1}\mathrm{C}}{(2n-1)(\mathrm{A}^2+k^2\mathrm{C}^2)^{2n-1}}\left[-(2n-1)\left\{(2n-2)\mathrm{A}^{2n-3}-k^2\mathrm{C}^2\,\frac{2n-2.2n-3.2n-4}{1.2.3}\mathrm{A}^{2n-5}+\ldots\right\}\right.$$
$$\left.+\frac{2.2n-1.\mathrm{A}}{\mathrm{A}^2+k^2\mathrm{C}^2}\left\{(2n-1)\mathrm{A}^{2n-2}-k^2\mathrm{C}^2\,\frac{2n-1.2n-2.2n-3}{1.2.3}\mathrm{A}^{2n-4}+\ldots\right\}\right]$$

$$=\frac{(-1)^n4k^2\mathrm{B}^{2n-1}\mathrm{C}}{(\mathrm{A}^2+k^2\mathrm{C}^2)^{2n}}\,\frac{(\mathrm{A}+ik\mathrm{C})^{2n}-(\mathrm{A}-ik\mathrm{C})^{2n}}{2ik\mathrm{C}}\text{ after a slight reduction and writing } i=\sqrt{-1}$$

$$=\frac{(-1)^n2k\mathrm{B}^{2n-1}}{i}\left[\frac{1}{(\mathrm{A}-ik\mathrm{C})^{2n}}-\frac{1}{(\mathrm{A}+ik\mathrm{C})^{2n}}\right].$$

Hence the whole coefficient as derived from all the terms in the expansion is

$$=\frac{2k}{i}\left[-\frac{\mathrm{B}}{(\mathrm{A}-ik\mathrm{C})^2}+\frac{\mathrm{B}^3}{(\mathrm{A}-ik\mathrm{C})^4}-\ldots+\frac{\mathrm{B}}{(\mathrm{A}+ik\mathrm{C})^2}-\frac{\mathrm{B}^3}{(\mathrm{A}+ik\mathrm{C})^4}+\ldots\right]$$

$$=-\frac{2k}{i}\left[\frac{\mathrm{B}}{\mathrm{B}^2+(\mathrm{A}-ik\mathrm{C})^2}-\frac{\mathrm{B}}{\mathrm{B}^2+(\mathrm{A}+ik\mathrm{C})^2}\right]$$

$$=-\frac{8k^2\mathrm{ABC}}{(\mathrm{A}^2+\mathrm{B}^2-k^2\mathrm{C}^2)^2+4k^2\mathrm{A}^2\mathrm{C}^2}$$

and this is the value of

$$\mathrm{E}(u_1)+\mathrm{E}(u_2)+\mathrm{E}(u_3)-\mathrm{E}(u_1+u_2+u_3).$$

Writing s, c, d respectively for sn$u$, cn$u$, dn$u$, the values of A, B and C are given by the equations

$$\mathrm{A}s_1+\mathrm{B}c_1+\mathrm{C}d_1=1$$
$$\mathrm{A}s_2+\mathrm{B}c_2+\mathrm{C}d_2=1$$
$$\mathrm{A}s_3+\mathrm{B}c_3+\mathrm{C}d_3=1$$

or writing

$$c_2d_3-c_3d_2+\ldots=\sigma$$
$$d_2s_3-d_3s_2+\ldots=\kappa$$
$$s_2c_3-s_3c_2+\ldots=\delta$$

$$\begin{vmatrix} s_1 & c_1 & d_1 \\ s_2 & c_2 & d_2 \\ s_3 & c_3 & d_3 \end{vmatrix}=\Delta$$

we have

$$\Delta\mathrm{A}=\sigma \qquad \Delta\mathrm{B}=\kappa \qquad \Delta\mathrm{C}=\delta$$

and then

$$\mathrm{E}(u_1)+\mathrm{E}(u_2)+\mathrm{E}(u_3)-\mathrm{E}(u_1+u_2+u_3)$$
$$=-\frac{8k^2\sigma\kappa\delta\Delta}{(\sigma^2+\kappa^2-k^2\delta^2)^2+4k^2\kappa^2\delta^2}.$$

As a verification of our formula assume

$$u_1=u_2=u_3$$

so that

$$x_1=x_2=x_3.$$

Then since the equation

$$\mathrm{A}s+\mathrm{B}c+\mathrm{C}d=1$$

has three equal roots, the values of A, B and C for our case will obviously be given by

$$\mathrm{A}s_1+\mathrm{B}c_1+\mathrm{C}d_1=1$$
$$\mathrm{A}-\mathrm{B}\frac{s_1}{c_1}-\mathrm{C}k^2\frac{s_1}{d_1}=0$$
$$\mathrm{B}\frac{1}{c_1{}^3}+\mathrm{C}k^2\frac{1}{d_1{}^3}=0;$$

for if we write $s_1+\xi$ for s_1 in the first, the coefficient of ξ must vanish, which condition gives the second equation; and similarly for the third. These last two equations give

$$\frac{\mathrm{A}}{k^2k'^2s_1{}^3}=\frac{\mathrm{B}}{-k^2c_1{}^3}=\frac{\mathrm{C}}{d_1{}^3}=\mu$$

and substituting in the first we find

$$\mu=\frac{1}{k'^2}.$$

From these we have at once

$$\frac{k'^4}{k^2}(A^2+B^2-k^2C^2)=k^2k'^4s^6-d^6+k^2c^6 \quad \text{(dropping subscripts)}$$

$$=-k'^2(1-3k^2s^4+2k^4s^6)$$

$$\frac{k'^2}{k^2}AC=s^3d^3.$$

Hence

$$\frac{k'^4}{k^4}\{(A^2+B^2-k^2C^2)^2+4k^2A^2C^2\}=(1-3k^2s^4+2k^4s^6)^2+4k^2s^6(1-k^2s^2)^3$$

$$=1-6k^2s^4+4(k^2+k^4)s^6-3k^2s^8$$

and we therefore have to verify that

$$3E(u)-E(3u)=\frac{8k^2s^3c^3d^3}{1-6k^2s^4+4(k^2+k^4)s^6-3k^4s^8}.$$

Now the ordinary addition formula for E is

$$E(u)+E(v)-E(u+v)=k^2\text{sn}u\,\text{sn}v\,\text{sn}(u+v)$$

so that

$$E(u)+E(2u)-E(3u)=k^2\text{sn}u\,\text{sn}2u\,\text{sn}3u$$

$$2E(u)-E(2u)=k^2\text{sn}^2u\,\text{sn}2u$$

and hence

$$3E(u)-E(3u)=k^2\text{sn}u\,\text{sn}2u(\text{sn}u+\text{sn}3u).$$

But

$$\text{sn}3u=\frac{3s-(4+4k^2)s^3+6k^2s^5-k^4s^9}{1-6k^2s^4+4(k^2+k^4)s^6-3k^4s^8}$$

or writing D for the denominator

$$D(\text{sn}u+\text{sn}3u)=4s\{1-(1+k^2)s^2+(k^2+k^4)s^4-k^4s^8\}$$

$$=4s(1-k^2s^4)(1-s^2)(1-k^2s^2)$$

$$=4s(1-k^2s^4)c^2d^2.$$

Moreover

$$\text{sn}2u=\frac{2scd}{1-k^2s^4}$$

so that

$$3E(u)-E(3u)=\frac{8k^2s^3c^3d^3}{D}$$

verifying the formula as required.

(iii.) In a similar way if we write

$$\Pi(\lambda,\ u_1)=\int_0^{x_1}\frac{dx}{(1-\lambda^2x^2)\sqrt{(1-x^2)(1-k^2x^2)}}$$

(so that $U=1, f(x)=1-\lambda^2x^2$) we shall obtain

$$\Pi(\lambda,\ u_1)+\Pi(\lambda,\ u_2)+\Pi(\lambda,\ u_3)+\Pi(\lambda,\ u_4)$$

$$=-\frac{\lambda}{2\sqrt{(1-\lambda^2)(k^2-\lambda^2)}}$$

$$\log\left[\frac{\{(A-\lambda)^2+(B\sqrt{1-\lambda^2}+C\sqrt{k^2-\lambda^2})^2\}\{(A+\lambda)^2+(B\sqrt{1-\lambda^2}-C\sqrt{k^2-\lambda^2})^2\}}{\{(A+\lambda)^2+(B\sqrt{1-\lambda^2}+C\sqrt{k^2-\lambda^2})^2\}\{(A-\lambda)^2+(B\sqrt{1-\lambda^2}-C\sqrt{k^2-\lambda^2})^2\}}\right]$$

the values of A, B, C being those which occur in the general case in (ii.).

Let $\lambda=k\,\mathrm{sn}a$ so as to introduce the third elliptic integral in the form used by JACOBI; then

$$\Pi(\lambda,\ u)=u+\frac{\mathrm{sn}a}{\mathrm{cn}a\mathrm{dn}a}\Pi(u,\ a)$$

$$=u+\frac{\lambda}{\sqrt{(1-\lambda^2)(k^2-\lambda^2)}}\Pi(u,\ a)$$

and the form of the theorem is now obviously as follows :—

$$\Pi(u_1,\ a)+\Pi(u_2,\ a)+\Pi(u_3,\ a)+\Pi(u_4,\ a)$$

$$=\tfrac{1}{2}\log\left[\frac{\{(A+ks')^2+(Bd'+Ckc')^2\}\{(A-ks')^2+(Bd'-Ckc')^2\}}{\{(A-ks')^2+(Bd'+Ckc')^2\}\{(A+ks')^2+(Bd'-Ckc')^2\}}\right]$$

where s', c', d' stand respectively for sn$a$, cn$a$, dn$a$.

12. *Example II.*

Take F_m and F_n as in example I., but now let

$$F_p\equiv Azy+(Bx+C)y+(Hx+D)z-Gx^2-Fx-1,$$

in effect the most general quadric relation. The eliminant X will be of the degree 8, and as there are seven arbitrary constants there will be only a single relation between the roots $x_1, x_2, \ldots, x_8$, which can be expressed in the form

$$\begin{vmatrix} c_1d_1 & s_1c_1 & s_1d_1 & s_1^2 & c_1 & d_1 & s_1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_8d_8 & s_8c_8 & s_8d_8 & s_8^2 & c_8 & d_8 & s_8 & 1 \end{vmatrix}=0.$$

Moreover, if we choose

$$\mathrm{B}=\mathrm{H}=\mathrm{F}=\mathrm{G}=0$$

$$-\mathrm{A}=\mathrm{C}=\mathrm{D}=1$$

the equation $\mathrm{X}=0$ is of the form

$$x^8=0$$

and we can therefore take zero for the lower limit in all our integrals. Hence we shall have

$$u_1+u_2+\ldots\qquad+u_8=0$$

where

$$x=\mathrm{sn}u$$

and the above relation will give $\mathrm{sn}(u_1+\ldots+u_7)$ in terms of the elliptic functions of $u_1,\ldots,u_7$.

Let us now find

$$\sum_{\mu=1}^{\mu=8}\int_0^{x_\mu}\sqrt{\frac{1-k^2x^2}{1-x^2}}dx\,;$$

write

$$\mathrm{B}x+\mathrm{C}=w_2;\qquad \mathrm{H}x+\mathrm{D}=w_1;\qquad 1+\mathrm{F}x+\mathrm{G}x^2=w_3;$$

then the right-hand side of the equation is

$$-\mathrm{C}_{\frac{1}{x}}\left[\frac{z}{y}\log\left\{\frac{w_3-zw_1-y(\mathrm{A}z+w_2)}{w_3-zw_1+y(\mathrm{A}z+w_2)}\right\}+\frac{z}{y}\log\left\{\frac{w_3+zw_1+y(\mathrm{A}z-w_2)}{w_3+zw_1-y(\mathrm{A}z-w_2)}\right\}\right].$$

On expansion, the n^{th} term gives

$$\frac{2}{2n-1}\mathrm{C}_{\frac{1}{x}}\left[z\frac{y^{2n-2}(\mathrm{A}z+w_2)^{2n-1}}{(w_3-zw_1)^{2n-1}}+z\frac{y^{2n-2}(\mathrm{A}z-w_2)^{2n-1}}{(w_3+zw_1)^{2n-1}}\right]$$

$$=\frac{2}{2n-1}\mathrm{C}_{\frac{1}{x}}\left[\frac{zy^{2n-2}}{(w_3{}^2-z^2w_1{}^2)^{2n-1}}\{(\mathrm{A}z+w_2)^{2n-1}(w_3+zw_1)^{2n-1}-(w_2-\mathrm{A}z)^{2n-1}(w_3-zw_1)^{2n-1}\}\right].$$

So far as the result is concerned, the expression within the inner bracket is

$$\{w_2w_3-k^2x^2w_1+z(\mathrm{A}w_3+w_1w_2)\}^{2n-1}\{w_2w_3-k^2x^2w_1-z(\mathrm{A}w_3+w_1w_2)\}^{2n-1}$$

$$=2z\{(2n-1)(w_2w_3-k^2x^2w_1)^{2n-2}(\mathrm{A}w_3+w_1w_2)+\ldots\}.$$

Now

$$w_2w_3-k^2x^2w_1=x^3\left[\mathrm{BG}-k^2\mathrm{H}+\frac{1}{x}(\mathrm{CG}+\mathrm{BF}-k^2\mathrm{D})\right]$$

$$=x^3\left(\lambda_1+\frac{1}{x}\lambda_2\right)\text{ say};$$

$$\mathrm{A}w_3+w_1w_2=x^2\left[\mathrm{AG}+\mathrm{BH}+\frac{1}{x}(\mathrm{AF}+\mathrm{BD}+\mathrm{CH})\right]$$

$$=x^2\left(\mu_1+\frac{1}{x}\mu_2\right)\text{ say ;}$$

$$w_3{}^2-z^2w_1{}^2=x^4(\mathrm{G}^2+k^2\mathrm{H}^2)\left\{1+2\frac{\mathrm{FG}+k^2\mathrm{DH}}{\mathrm{G}^2+k^2\mathrm{H}^2}\,\frac{1}{x}\right\}$$

$$=x^4\rho_1\left(1+\frac{1}{x}\rho_2\right)\text{ say.}$$

Thus the n^{th} term gives

$$\frac{4(-1)^nk^2}{2n-1}\,\frac{1}{\rho_1{}^{2n-1}}\mathrm{C}_{\frac{1}{x}}\left(1-\frac{2n-1}{x}\rho_2\right)\left[(2n-1)\left(\lambda_1{}^{2n-2}+\frac{2n-2}{x}\lambda_1{}^{2n-3}\lambda_2\right)\left(\mu_1+\frac{1}{x}\mu_2\right)\right.$$
$$\left.-k^2\frac{2n-1.2n-2.2n-3}{1.2.3}\left(\lambda_1{}^{2n-4}+\frac{2n-4}{x}\lambda_1{}^{2n-5}\lambda_2\right)\left(\mu_1{}^3+\frac{3}{x}\mu_1{}^2\mu_2\right)+\ldots\right]$$

$$=\frac{-4(-1)^nk^2}{\rho_1{}^{2n-1}}\rho_2\left\{(2n-1)\lambda_1{}^{2n-2}\mu_1-k^2\frac{2n-1.2n-2.2n-3}{1.2.3}\lambda_1{}^{2n-4}\mu_1{}^3+\ldots\right\}$$
$$+\frac{4(-1)^nk^2}{\rho_1{}^{2n-1}}\left\{\lambda_1{}^{2n-2}\mu_2-k^2\frac{2n-2.2n-3}{1.2}\lambda_1{}^{2n-4}\mu_1{}^2\mu_2+\ldots\right.$$
$$\left.+(2n-2)\lambda_1{}^{2n-3}\lambda_2\mu_1-k^2\frac{2n-2.2n-3.2n-4}{1.2.3}\lambda_1{}^{2n-5}\lambda_2\mu_1{}^3+\ldots\right\}$$

$$=\frac{4(-1)^nk^2}{\rho_1{}^{2n-1}}\left[-\frac{\rho_2}{2ik}\{(\lambda_1+ik\mu_1)^{2n-1}-(\lambda_1-ik\mu_1)^{2n-1}\}+\frac{\mu_2}{2}\{(\lambda_1+ik\mu_1)^{2n-2}+(\lambda_1-ik\mu_1)^{2n-2}\}\right.$$
$$\left.+\frac{\lambda_2}{2ik}\{(\lambda_1+ik\mu_1)^{2n-2}-(\lambda_1-ik\mu_1)^{2n-2}\}\right]$$

i denoting $\sqrt{-1}$. Hence summing up for all the terms and reducing we have the whole coefficient equal to

$$-\frac{4k^2\{(\rho_1{}^2-k^2\mu_1{}^2)(\mu_2\rho_1-\rho_1\rho_2\mu_1)+\lambda_1{}^2(\mu_2\rho_1+\rho_1\rho_2\mu_1)-2\lambda_1\lambda_2\mu_1\rho_1\}}{(\rho_1{}^2+\lambda_1{}^2-k^2\mu_1{}^2)^2+4k^2\lambda_1{}^2\mu_1{}^2}.$$

Now

$$\begin{array}{ll}\rho_1=\mathrm{G}^2+k^2\mathrm{H}^2 & \mu_1=\mathrm{AG}+\mathrm{BH}\\ \rho_1\rho_2=2(\mathrm{FG}+k^2\mathrm{DH}), & \mu_2=\mathrm{AF}+\mathrm{BD}+\mathrm{CH}\\ \lambda_1=\mathrm{BG}-k^2\mathrm{H}, & \lambda_2=\mathrm{CG}+\mathrm{BF}-k^2\mathrm{D}\end{array}$$

and the values of A, B, C, D, H, F, G are determined by the seven equations

$$\mathrm{A}c_\mu d_\mu+\mathrm{B}s_\mu c_\mu+\mathrm{H}s_\mu d_\mu-\mathrm{G}s_\mu{}^2+\mathrm{C}c_\mu+\mathrm{D}d_\mu-\mathrm{F}s_\mu=1$$

$(\mu=1, 2, \ldots, 7)$; and therefore the above is the value of

$$\sum_{\mu=1}^{\mu=7} \mathrm{E}(u_\mu) - \mathrm{E}\left(\sum_{\mu=1}^{\mu=7} u_\mu\right)$$

expressed in terms of the functions of the u's.

The evaluation of the corresponding expression for the sum of the third elliptic integrals presents no difficulty.

SECTION II.

Abelian functions, after WEIERSTRASS.

13. The theory of these functions is detailed in a paper by WEIERSTRASS in CRELLE'S Journal, t. lii., but such formulæ as may be necessary in what follows will be proved. Let

$$\left.\begin{aligned} y^2-\mathrm{P}(x) &\equiv y^2-(x-a_1)(x-a_2)\ldots(x-a_\rho)=0 \\ z^2-\mathrm{Q}(x) &\equiv z^2-(x-a_{\rho+1})(x-a_{\rho+2})\ldots(x-a_{2\rho+1})=0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\theta \equiv \mathrm{M}y+\mathrm{N}z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where M is of the degree ρ in x, N of $\rho-1$; say

$$\left.\begin{aligned} \mathrm{M} &\equiv x^\rho+\mathrm{M}_1x^{\rho-1}+\ldots+\mathrm{M}_{\rho-1}x+\mathrm{M}_\rho \\ \mathrm{N} &\equiv \qquad \mathrm{N}_1x^{\rho-1}+\ldots+\mathrm{N}_{\rho-1}x+\mathrm{N}_\rho \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Then the equation for the roots x being

$$\mathrm{M}^2y^2-\mathrm{N}^2z^2=0$$

is of the degree 3ρ and involves 2ρ arbitrary constants; thus there must be ρ relations among the roots. Let these roots be denoted by $x_1, x_2, \ldots, x_\rho$; $\xi_1, \xi_2, \ldots, \xi_\rho$; $p_1, p_2, \ldots, p_\rho$; so that we may consider the ρ p's as given in terms of the x's and ξ's by the ρ relations which might be exhibited in a determinantal form. Write

$$\mathrm{R}(x)=\mathrm{P}(x)\mathrm{Q}(x)$$

and let

$$u_\mu=\tfrac{1}{2}\sum_{\lambda=1}^{\lambda=\rho}\int_{a_\lambda}^{x_\lambda}\frac{\mathrm{P}(x)dx}{(x-a_\mu)\sqrt{\mathrm{R}(x)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in which μ has in succession the values 1, 2, . . . , ρ as also in

$$v_\mu=\tfrac{1}{2}\sum_{\lambda=1}^{\lambda=\rho}\int_{a_\lambda}^{\xi_\lambda}\frac{\mathrm{P}(x)dx}{(x-a_\mu)\sqrt{\mathrm{R}(x)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and in

$$w_\mu = \tfrac{1}{2}\sum_{\lambda=1}^{\lambda=\rho}\int_{a_\lambda}^{p_\lambda}\frac{P(x)dx}{(x-a_\mu)\sqrt{R(x)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Now the constants in M and N can be so chosen that the roots of the equation

$$M^2y^2 - N^2z^2 = 0$$

shall be $a_1, a_2, \ldots, a_\rho$ each occurring thrice; for the conditions are that any one, say a_λ, shall satisfy the three equations

$$M^2y^2 - N^2z^2 = 0$$

$$M^2y\frac{dy}{dx} + y^2M\frac{dM}{dx} - N^2z\frac{dz}{dx} - z^2N\frac{dN}{dx} = 0$$

$$MT + y^2\left(\frac{dM}{dx}\right)^2 - NU - z^2\left(\frac{dN}{dx}\right)^2 = 0$$

where T, U are functions of x. In order that a_λ may satisfy the first of these

$$N = 0$$

when $x = a_\lambda$; that it may satisfy the second the additional condition is that

$$M = 0$$

and therefore M will contain $x - a_\lambda$ as a factor. Moreover, y^2 contains $x - a_\lambda$ as a factor and therefore in M^2y^2 we shall have $x - a_\lambda$ raised to the third power; and therefore if we expand N in ascending powers of $x - a_\lambda$ the first term is of the order

$$(x - a_\lambda)^{\frac{3}{2}}.$$

Hence $\left(\frac{dN}{dx}\right)^2$ has $A(x - a_\lambda)$ as its first term and so vanishes when $x = a_\lambda$. But this is the additional condition that $x = a_\lambda$ should satisfy the third equation, and being satisfied it proves that a_λ may occur as a triple root provided

$$M = 0 \qquad N = 0$$

when $x = a_\lambda$. That is to say, two conditions are necessary for each root, or 2ρ in all; but as there are 2ρ disposable constants these can all be satisfied and so the truth of the proposition is established. But as N is only of the degree $\rho - 1$ in x, while it has to vanish for ρ values, it must be identically zero; and we choose $M = y^2$, so that the equation is $M^2y^2 = 0$ which is obvious beforehand.

14. Applying now the general theorem from § 6 we have

$$\sum_{\lambda=1}^{\lambda=\rho}\left\{\int^{x_\lambda}\frac{P(x)dx}{(x-a_\mu)\sqrt{R(x)}} + \int^{\xi_\lambda}\frac{P(x)dx}{(x-a_\mu)\sqrt{R(x)}} + \int^{p_\lambda}\frac{P(x)dx}{(x-a_\mu)\sqrt{R(x)}}\right\} = \text{constant}$$

since $\frac{\mathrm{P}(x)}{x-a_\mu}$ is an integral function of x, and when $\frac{\mathrm{P}(x)}{(x-a_\mu)\sqrt{\mathrm{R}(x)}}$ is expanded in descending powers of x the highest index of x is $-\frac{3}{2}$. We have shown that $a_1, a_2, \ldots, a_\lambda$ may be regarded as triple roots of the equation for the roots, and thus we may take as the constant

$$3\sum_{\lambda=1}^{\lambda=\rho}\int^{a_\lambda}\frac{\mathrm{P}(x)dx}{(x-a_\mu)\sqrt{\mathrm{R}(x)}}.$$

Hence

$$\sum_{\lambda=1}^{\lambda=\rho}\left\{\int_{a_\lambda}^{x_\lambda}\frac{\mathrm{P}(x)dx}{(x-a_\mu)\sqrt{\mathrm{R}(x)}}+\int_{a_\lambda}^{\xi_\lambda}\frac{\mathrm{P}(x)dx}{(x-a_\mu)\sqrt{\mathrm{R}(x)}}+\int_{a_\lambda}^{p_\lambda}\frac{\mathrm{P}(x)dx}{(x-a_\mu)\sqrt{\mathrm{R}(x)}}\right\}=0$$

or

$$u_\mu+v_\mu+w_\mu=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Now, by WEIERSTRASS's theory, given values of $u_1, u_2, \ldots, u_\rho$ imply unique values of $x_1, x_2, \ldots, x_\rho$ which are, in fact, the roots of an equation of the ρ^{th} degree whose coefficients are single-valued functions of $u_1, u_2, \ldots, u_\rho$. Every symmetrical function of $x_1, \ldots, x_\rho$ can therefore be expressed as a function $u_1, u_2, \ldots, u_\rho$, but in particular

$$(a_t-x_1)(a_t-x_2)\ldots(a_t-x_\rho)$$

(t being any of the integers $1, 2, \ldots, 2\rho+1$) is the perfect square of such a function. Write

$$\phi(x) \quad =(x-x_1)(x-x_2)\ldots(x-x_\rho) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\left.\begin{array}{l} -\mathrm{Q}(a_r) \quad =l_r \quad (r=1, 2, \ldots, \rho) \\ \mathrm{P}(a_{\rho+s})=l_{\rho+s}(s=1, 2, \ldots, \rho+1) \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

then WEIERSTRASS defines

$$l_r al_r^2=\phi(a_r). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

for all values of r included in $1, 2, \ldots, 2\rho+1$. It is easy to verify that $x_1, x_2, \ldots, x_\rho$ are the roots of

$$\sum_{r=1}^{r=\rho}\left\{\frac{l_r al_r^2}{(a_r-x)\mathrm{P}'(a_r)}\right\}=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

for there are obviously ρ roots, and in order that x_1 may be one of these we must have

$$\sum_{r=1}^{r=\rho}\frac{(a_r-x_2)(a_r-x_3)\ldots(a_r-x_\rho)}{(a_r-a_1)(a_r-a_2)\ldots(a_r-a_\rho)}=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{A}).$$

By a known theorem of ABEL's we have

$$\Sigma\frac{z^s}{\frac{d\chi}{dz}}=0 \text{ or } 1,$$

according as $s<$ or $=\rho-1$, the summation being extended over the ρ roots of $\chi(x)=0$; and therefore

$$\sum_{r=1}^{r=\rho}\frac{a_r^{\rho-1}}{\mathrm{P}'(a_r)}=0$$

$$\sum_{r=1}^{r=\rho}\frac{a_r^{s}}{\mathrm{P}'(a_r)}=0 \text{ if } s<\rho-1\geqq 0.$$

Thus the left-hand side of (A)

$$\begin{aligned}&=\sum_{r=1}^{r=\rho}\left\{\frac{a_r^{\rho-1}}{\mathrm{P}'(a_r)}-(x_2+\ldots+x_\rho)\frac{a_r^{\rho-2}}{\mathrm{P}'(a_r)}+\ldots\right\}\\&=1\end{aligned}$$

all the terms disappearing except the first and so verifying (A) and proving that $x_1, x_2, \ldots, x_\rho$ are the roots of (11).

15. Taking now our set of integrals u we have

$$\left.\begin{array}{l}0=\frac{1}{2}\sum\limits_{\lambda=1}^{\lambda=\rho}\dfrac{\mathrm{P}(x_\lambda)}{(x_\lambda-a_1)\sqrt{\mathrm{R}(x_\lambda)}}\dfrac{\partial x_\lambda}{\partial u_\mu}\\0=\frac{1}{2}\sum\limits_{\lambda=1}^{\lambda=\rho}\dfrac{\mathrm{P}(x_\lambda)}{(x_\lambda-a_2)\sqrt{\mathrm{R}(x_\lambda)}}\dfrac{\partial x_\lambda}{\partial u_\mu}\\\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\1=\frac{1}{2}\sum\limits_{\lambda=1}^{\lambda=\rho}\dfrac{\mathrm{P}(x_\lambda)}{(x_\lambda-a_\mu)\sqrt{\mathrm{R}(x_\lambda)}}\dfrac{\partial x_\lambda}{\partial u_\mu}\\\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\0=\frac{1}{2}\sum\limits_{\lambda=1}^{\lambda=\rho}\dfrac{\mathrm{P}(x_\lambda)}{(x_\lambda-a_\rho)\sqrt{\mathrm{R}(x_\lambda)}}\dfrac{\partial x_\lambda}{\partial u_\mu}\end{array}\right\}\quad\ldots\ldots\ldots\quad(12).$$

Multiply these respectively by $\frac{l_1al_1^2}{\mathrm{P}'(a_1)}, \frac{l_2al_2^2}{\mathrm{P}'(a_2)}, \ldots$ and add; then, in virtue of equation (11),

$$\frac{l_\mu al_\mu^2}{\mathrm{P}'(a_\mu)}=-\frac{1}{2}\sum_{\lambda=1}^{\lambda=\rho}\frac{\mathrm{P}(x_\lambda)}{\sqrt{\mathrm{R}(x_\lambda)}}\frac{\partial x_\lambda}{\partial u_\mu}$$

so that if we write

$$\mathrm{U}=\frac{1}{2}\sum_{\lambda=1}^{\lambda=\rho}\int_{a_\lambda}^{x_\lambda}\frac{\mathrm{P}(x)dx}{\sqrt{\mathrm{R}(x)}}\quad\ldots\ldots\ldots\quad(13)$$

we have

$$\frac{l_\mu al_\mu^2}{\mathrm{P}'(a_\mu)}=-\frac{\partial\mathrm{U}}{\partial u_\mu}\quad\ldots\ldots\ldots\quad(14).$$

U is obviously a symmetric function of the x's, and is therefore a function of the u's.

16. Again solving the equations (12) regarded as giving $\frac{\delta x_\lambda}{\delta u_\mu}$ for different values of λ, we have*

$$\tfrac{1}{2}\frac{\delta x_\lambda}{\delta u_\mu}=-\frac{\sqrt{\mathrm{R}(x_\lambda)}}{\phi'(x_\lambda)}\,\frac{\phi(a_\mu)}{\mathrm{P}'(a_\mu)}\,\frac{1}{x_\lambda-a_\mu}$$

$$=-\frac{\sqrt{\mathrm{R}(x_\lambda)}}{\phi'(x_\lambda)}\,\frac{l_\mu al_\mu{}^2}{(x_\lambda-a_\mu)\mathrm{P}'(a_\mu)}\quad .\ .\ .\ .\ .\ .\ .\ .\ .\quad (15).$$

Therefore

$$\tfrac{1}{2}\sum_{\mu=1}^{\mu=\rho}\left[\frac{\phi'(x_\lambda)}{\sqrt{\mathrm{R}(x_\lambda)}}\,\frac{\delta x_\lambda}{\delta u_\mu}\right]=-\sum_{\mu=1}^{\mu=\rho}\frac{l_\mu al_\mu{}^2}{(x_\lambda-a_\mu)\mathrm{P}'(a_\mu)}$$

$$=1 \text{ by (11)}$$

and

$$\sum_{\lambda=1}^{\lambda=\rho}\left\{\frac{\sqrt{\mathrm{R}(x_\lambda)}}{\phi'(x_\lambda)}\,\frac{1}{x_\lambda-a_s}\right\}=\tfrac{1}{2}\sum_{\lambda=1}^{\lambda=\rho}\sum_{\mu=1}^{\mu=\rho}\frac{1}{x_\lambda-a_s}\,\frac{\delta x_\lambda}{\delta u_\mu}$$

$$=\tfrac{1}{2}\sum_{\mu=1}^{\mu=\rho}\frac{\delta}{\delta u_\mu}\left\{\log\prod_{\lambda=1}^{\lambda=\rho}(a_s-x_\lambda)\right\}$$

$$=\tfrac{1}{2}\sum_{\mu=1}^{\mu=\rho}\frac{\delta}{\delta u_\mu}\log\phi(a_s)$$

$$=\frac{1}{al_s}\sum_{\mu=1}^{\mu=\rho}\frac{\delta al_s}{\delta u_\mu}\quad .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\quad (16')$$

where s may be any of the integers 1, 2, . . . , $2\rho+1$.

Writing

$$\sum_{\mu=1}^{\mu=\rho}\frac{\delta al_s}{\delta u_\mu}=\overline{al}_s\quad .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\quad (17)$$

so that (16′) becomes

$$\sum_{\lambda=1}^{\lambda=\rho}\left\{\frac{\sqrt{\mathrm{R}(x_\lambda)}}{\phi'(x_\lambda)}\,\frac{1}{x_\lambda-a_s}\right\}=\frac{\overline{al}_s}{al_s}\quad .\ .\ .\ .\ .\ .\ .\ .\ .\ .\quad (16)$$

Weierstrass defines

$$al_{r,s}=\frac{al_r\overline{al}_s-al_s\overline{al}_r}{a_r-a_s}$$

$$=\frac{al_r al_s}{a_r-a_s}\left[\frac{\overline{al}_s}{al_s}-\frac{\overline{al}_r}{al_r}\right]$$

$$=-\sum_{\lambda=1}^{\lambda=\rho}\left\{\frac{\sqrt{\mathrm{R}(x_\lambda)}}{\phi'(x_\lambda)}\,\frac{al_r al_s}{(x_\lambda-a_r)(x_\lambda-a_s)}\right\}\quad .\ .\ .\ .\ .\ .\ .\ .\quad (18)$$

where r, s must be different from each other, but otherwise may be any of the integers

---

* *Cf.* Scott's 'Determinants,' c. ix., §§ 11, 12.

$1, 2, \ldots, 2\rho+1$. Evidently $al_{r,s}=al_{s,r}$, and there are therefore $\rho(2\rho+1)$ functions $al_{r,s}$; these, together with the $2\rho+1$ functions al_s, are the functions of the theory. (They are, of course, not all independent; the complete system of relations among them may be found in the fifth section of the first chapter of the memoir already quoted.)

Further

$$\frac{1}{al_r}\frac{\partial al_r}{\partial u_s}=-\tfrac{1}{2}\sum_{\lambda=1}^{\lambda=\rho}\frac{1}{a_r-x_\lambda}\frac{\partial x_\lambda}{\partial u_s}$$

$$=\sum_{\lambda=1}^{\lambda=\rho}\frac{\sqrt{\mathrm{R}(x_\lambda)}}{\phi'(x_\lambda)}\frac{l_s al_s^2}{(x_\lambda-a_s)\mathrm{P}'(a_s)}\frac{1}{(a_r-x_\lambda)}$$

$$=\frac{l_s al_s^2}{\mathrm{P}'(a_s)}\frac{al_{r,s}}{al_r al_s}$$

and

$$\frac{\partial al_r}{\partial u_s}=\frac{l_s}{\mathrm{P}'(a_s)}al_s al_{r,s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

in which

s may have any value $1, 2, \ldots, \rho$
r ,, ,, $1, 2, \ldots, 2\rho+1$,

but r, s may not be equal. If $r\leqq\rho$, this serves as a verification of (14).

Again, since $x_1, x_2, \ldots, x_\rho$ are the roots of (11),

$$\sum_{s=1}^{s=\rho}\frac{l_s al_s^2}{(x-a_s)\mathrm{P}'(a_s)}+1=\frac{(x-x_1)(x-x_2)\ldots(x-x_\rho)}{(x-a_1)(x-a_2)\ldots(x-a_\rho)}=\frac{\phi(x)}{\mathrm{P}(x)}.$$

In this write $x=a_{\rho+r}$, $(r=1, 2, \ldots, \rho+1)$; then

$$al^2_{\rho+r}=\frac{\phi(a_{\rho+r})}{l_{\rho+r}}$$

$$=1-\sum_{s=1}^{s=\rho}\frac{l_s al_s^2}{(a_s-a_{\rho+r})\mathrm{P}'(a_s)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

which expresses $\rho+1$ functions $al_{\rho+r}$ each in terms of the ρ functions $al_1, al_2, \ldots, al_\rho$. By (20) and (14) we have

$$al^2_{\rho+r}=1+\sum_{s=1}^{s=\rho}\frac{1}{a_s-a_{\rho+r}}\frac{\partial \mathrm{U}}{\partial u_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21).$$

[17. A simpler form can be given to this equation by the introduction of a series of $\rho+1$ new variables provisionally given by

$$\left.\begin{aligned} u_1 &= \sum_{r=1}^{r=\rho+1} \frac{u_{\rho+r}}{a_1 - a_{\rho+r}} \\ u_2 &= \sum_{r=1}^{r=\rho+1} \frac{u_{\rho+r}}{a_2 - a_{\rho+r}} \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ u_\rho &= \sum_{r=1}^{r=\rho+1} \frac{u_{\rho+r}}{a_\rho - a_{\rho+r}} \end{aligned}\right\}$$

These new u's are not fully determined: as the remaining equation necessary to determine them assume

$$f(u_{\rho+1}, \ldots, u_{2\rho+1}) = 0.$$

When substitution is made in U for $u_1, \ldots, u_\rho$, U will be a function of $u_{\rho+1}, \ldots, u_{2\rho+1}$; and we shall have

$$\sum_{r=1}^{r=\rho+1} \frac{\delta \mathrm{U}}{\delta u_{\rho+r}} \delta u_{\rho+r} = \sum_{s=1}^{s=\rho} \frac{\delta \mathrm{U}}{\delta u_s} \delta u_s$$

$$= \sum_{s=1}^{s=\rho} \sum_{r=1}^{r=\rho+1} \frac{1}{a_s - a_{\rho+r}} \frac{\delta \mathrm{U}}{\delta u_s} \delta u_{\rho+r}$$

and from the $(\rho+1)^{\text{th}}$ equation giving the new u's

$$\sum_{r=1}^{r=\rho+1} \frac{\delta f}{\delta u_{\rho+r}} \delta u_{\rho+r} = 0.$$

Then by the principle of indeterminate multipliers

$$\frac{\delta \mathrm{U}}{\delta u_{\rho+r}} - \sum_{s=1}^{s=\rho} \frac{1}{a_s - a_{\rho+r}} \frac{\delta \mathrm{U}}{\delta u_s} = \lambda \frac{\delta f}{\delta u_{\rho+r}}$$

for all the $\rho+1$ values of r. Multiply these $\rho+1$ equations by $u_{\rho+1}, u_{\rho+2}, \ldots$ respectively and add; then

$$\sum_{r=1}^{r=\rho+1} u_{\rho+r} \frac{\delta \mathrm{U}}{\delta u_{\rho+r}} - \sum_{s=1}^{s=\rho} u_s \frac{\delta \mathrm{U}}{\delta u_s} = \lambda \sum_{r=1}^{r=\rho+1} u_{\rho+r} \frac{\delta f}{\delta u_{\rho+r}}.$$

Let the part of U which is of the order m in the u_s's $(s \leqq \rho)$ be denoted by U_m; then when expressed in terms of the $u_{\rho+r}$'s it still remains the term of order m, so that

$$\sum_{r=1}^{r=\rho+1} u_{\rho+r} \frac{\delta \mathrm{U}_m}{\delta u_{\rho+r}} = m\mathrm{U}_m = \sum_{s=1}^{s=\rho} u_s \frac{\delta \mathrm{U}_m}{\delta u_s}$$

and summing up for the terms of all orders

$$\sum_{r=1}^{r=\rho+1} u_{\rho+r} \frac{\delta \mathrm{U}}{\delta u_{\rho+r}} = \sum_{s=1}^{s=\rho} u_s \frac{\delta \mathrm{U}}{\delta u_s}$$

and therefore from the above

$$\lambda \sum_{r=1}^{r=\rho+1} u_{\rho+r} \frac{\partial f}{\partial u_{\rho+r}} = 0,$$

equivalent to one of the two equations

$$\lambda = 0$$

or

$$\sum_{r=1}^{r=\rho+1} u_{\rho+r} \frac{\partial f}{\partial u_{\rho+r}} = 0.$$

The latter, taken with the equation

$$f(u_{\rho+1}, \ldots, u_{2\rho+1}) = 0,$$

implies that there is a homogeneous relation between the quantities $u_{\rho+r}$; this we may reject. The former leaves f arbitrary or non-existent, and so there would be only ρ equations to determine $\rho+1$ quantities, a difficulty, however, obviable at any time by assigning some new equation to make up the requisite number; but $\lambda=0$ simplifies the resulting equations in which it occurs, and therefore this is selected. Let us assume as our new equation

$$\frac{v}{b} = \frac{u_{\rho+1}}{b-a_{\rho+1}} + \frac{u_{\rho+2}}{b-a_{\rho+2}} + \ldots + \frac{u_{2\rho+1}}{b-a_{2\rho+1}}$$

where v is a quantity which may have a definite value assigned to it at any time, if desired. Thus we have

$$\frac{\partial \mathrm{U}}{\partial u_{\rho+r}} = \sum_{s=1}^{s=\rho} \frac{1}{a_s - a_{\rho+r}} \frac{\partial \mathrm{U}}{\partial u_s}$$

and therefore

$$al^2_{\rho+r} = 1 + \frac{\partial \mathrm{U}}{\partial u_{\rho+r}} \quad \ldots \ldots \ldots \quad (14')$$

similar in form to (14).

18. Let us obtain the new u's explicitly from the above equations. Writing

$$g(z) = (z-b)\mathrm{P}(z)$$

we have*

$$u_{\rho+r} = -\frac{g(a_{\rho+r})}{\mathrm{Q}'(a_{\rho+r})} \left[\frac{\mathrm{Q}(a_1)}{g'(a_1)} \frac{u_1}{a_1 - a_{\rho+r}} + \ldots + \frac{\mathrm{Q}(a_\rho)}{g'(a_\rho)} \frac{u_\rho}{a_\rho - a_{\rho+r}} + \frac{\mathrm{Q}(b)}{g'(b)} \frac{\frac{v}{b}}{b - a_{\rho+r}} \right].$$

Now make b infinite, so that the assumed equation takes the form

$$u_{\rho+1} + u_{\rho+2} + \ldots + u_{2\rho+1} = v$$

and

---

* *Cf.* SCOTT's 'Determinants,' *l.c.*

$$\frac{g(a_{\rho+r})}{g'(a_s)}=\frac{\mathrm{P}(a_{\rho+r})}{\mathrm{P}'(a_s)}$$

$$\frac{g(a_{\rho+r})}{b-a_{\rho+r}}=-\mathrm{P}(a_{\rho+r})$$

$$\frac{\mathrm{Q}(b)}{bg'(b)}=1$$

so that

$$-u_{\rho+r}=\sum_{s=1}^{s=\rho}\left\{\frac{u_s}{a_s-a_{\rho+r}}\,\frac{\mathrm{P}(a_{\rho+r})}{\mathrm{Q}'(a_{\rho+r})}\,\frac{\mathrm{Q}(a_s)}{\mathrm{P}'(a_s)}\right\}-\frac{\mathrm{P}(a_{\rho+r})}{\mathrm{Q}'(a_{\rho+r})}v.$$

As a verification of these values we may deduce (14) from (14′) as follows :—

$$\sum_{s=1}^{s=\rho}\frac{\delta\mathrm{U}}{\delta u_s}\delta u_s=d\mathrm{U}=\sum_{r=1}^{r=\rho+1}\frac{\delta\mathrm{U}}{\delta u_{\rho+r}}\delta u_{\rho+r}$$

$$=\sum_{r=1}^{r=\rho+1}\sum_{s=1}^{s=\rho}\frac{\delta\mathrm{U}}{\delta u_{\rho+r}}\,\frac{l_sl_{\rho+r}}{\mathrm{P}'(a_s)\mathrm{Q}'(a_{\rho+r})}\,\frac{\delta u_s}{a_s-a_{\rho+r}}+\sum_{r=1}^{r=\rho+1}\frac{l_{\rho+r}}{\mathrm{Q}'(a_{\rho+r})}\,\frac{\delta\mathrm{U}}{\delta u_{\rho+r}}\delta v.$$

Now the quantities δu_s, δv are independent ; hence we must have

$$\frac{\delta\mathrm{U}}{\delta u_s}=\sum_{r=1}^{r=\rho+1}\frac{l_sl_{\rho+r}}{\mathrm{P}'(a_s)\mathrm{Q}'(a_{\rho+r})}\,\frac{1}{a_s-a_{\rho+r}}\,\frac{\delta\mathrm{U}}{\delta u_{\rho+r}},$$

$$0=\sum_{r=1}^{r=\rho+1}\frac{l_{\rho+r}}{\mathrm{Q}'(a_{\rho+r})}\,\frac{\delta\mathrm{U}}{\delta u_{\rho+r}}.$$

Taking the second of these, we have

$$\frac{\delta\mathrm{U}}{\delta u_{\rho+r}}=-1+al^2_{\rho+r}$$

$$=-1+\frac{\phi(a_{\rho+r})}{l_{\rho+r}}$$

and therefore

$$\sum_{r=1}^{r=\rho+1}\frac{l_{\rho+r}}{\mathrm{Q}'(a_{\rho+r})}\,\frac{\delta\mathrm{U}}{\delta u_{\rho+r}}=-\sum_{r=1}^{r=\rho+1}\frac{l_{\rho+r}}{\mathrm{Q}'(a_{\rho+r})}+\sum_{r=1}^{r=\rho+1}\frac{\phi(a_{\rho+r})}{\mathrm{Q}'(a_{\rho+r})}$$

$$=-1+1=0$$

by the theorem already quoted in the verification of equation (11). For the first summation we have

$$\sum_{r=1}^{r=\rho+1}\frac{l_s}{\mathrm{P}'(a_s)}\,\frac{l_{\rho+r}}{\mathrm{Q}'(a_{\rho+r})}\,\frac{1}{a_s-a_{\rho+r}}\,\frac{\delta\mathrm{U}}{\delta u_{\rho+r}}$$

$$=\frac{l_s}{\mathrm{P}'(a_s)}\left[-\sum_{r=1}^{r=\rho+1}\frac{l_{\rho+r}}{\mathrm{Q}'(a_{\rho+r})}\,\frac{1}{a_s-a_{\rho+r}}+\sum_{r=1}^{r=\rho+1}\frac{l_{\rho+r}al^2_{\rho+r}}{\mathrm{Q}'(a_{\rho+r})(a_s-a_{\rho+r})}\right].$$

The first term inside the bracket is the expansion in partial fractions of

$$-\frac{\mathrm{P}(a_s)}{\mathrm{Q}(a_s)}$$

and is therefore zero since $s \leqq \rho$; the second is

$$\sum_{r=1}^{r=\rho+1}\left\{\frac{\phi(a_{\rho+r})}{\mathrm{Q}'(a_{\rho+r})}\,\frac{1}{a_s-a_{\rho+r}}\right\}$$

$$=\frac{\phi(a_s)}{\mathrm{Q}(a_s)}=-al_s^2 \quad \text{by definition}$$

so that the equation with which we began leads to

$$\frac{\delta\mathrm{U}}{\delta u_s}=-\frac{l_s}{\mathrm{P}'(a_s)}al_s^2$$

that is, to equation (14).]

19. Now let

$$\left.\begin{aligned}\mathrm{V}&=\tfrac{1}{2}\sum_{\lambda=1}^{\lambda=r}\int_{a_\lambda}^{\xi_\lambda}\frac{\mathrm{P}(x)}{\sqrt{\mathrm{R}(x)}}dx\\ \mathrm{W}&=\tfrac{1}{2}\sum_{\lambda=1}^{\lambda=\rho}\int_{a_\lambda}^{p_\lambda}\frac{\mathrm{P}(x)}{\sqrt{\mathrm{R}(x)}}dx\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (13')$$

so that V stands to the v's and W to the w's in exactly the same relation as U to the u's.

Applying now the theorem in § 6 we have

$$\mathrm{U}+\mathrm{V}+\mathrm{W}=-\mathrm{C}_{\frac{1}{x}}\tfrac{1}{2}\frac{\mathrm{P}(x)}{\sqrt{\mathrm{R}(x)}}\log\left(\frac{\mathrm{M}y+\mathrm{N}z}{\mathrm{M}y-\mathrm{N}z}\right)$$

$$=-\mathrm{C}_{\frac{1}{x}}\frac{y}{z}\left[\frac{\mathrm{N}z}{\mathrm{M}y}+\tfrac{1}{3}\left(\frac{\mathrm{N}z}{\mathrm{M}y}\right)^3+\ldots\right].$$

The n^{th} term in this series gives

$$-\frac{1}{2n-1}\mathrm{C}_{\frac{1}{x}}\left(\frac{\mathrm{N}}{\mathrm{M}}\right)^{2n-1}\left(\frac{z}{y}\right)^{2n-2}$$

$$=-\frac{1}{2n-1}\mathrm{C}_{\frac{1}{x}}\left\{\frac{\mathrm{N}_1^{2n-1}}{x^n}+\text{higher powers of }\frac{1}{x}\right\}$$

so that nothing is contributed except by the first term, and we have

$$\mathrm{U}+\mathrm{V}+\mathrm{W}=-\mathrm{N}_1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (22).$$

The 2ρ quantities $\mathrm{N}_1, \mathrm{N}_2, \ldots, \mathrm{N}_\rho, \mathrm{M}_1, \ldots, \mathrm{M}_\rho$ are determined by the equation

$$\mathrm{N}_1x^{\rho-1}z+\mathrm{N}_2x^{\rho-2}z+\ldots+\mathrm{N}_\rho z+\mathrm{M}_1x^{\rho-1}y+\ldots+\mathrm{M}_\rho y=-x^\rho y$$

which is satisfied by the 2ρ values of x, viz.: $x_1, x_2, \ldots, x_\rho, \xi_1, \xi_2, \ldots, \xi_\rho$; and therefore

$$\mathrm{N}_1 \begin{vmatrix} x_1^{\rho-1}z_1, x_1^{\rho-2}z_1, \ldots, y_1 \\ x_2^{\rho-1}z_2, x_2^{\rho-2}z_2, \ldots, y_2 \\ \cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot \\ x_\rho^{\rho-1}z_\rho, x_\rho^{\rho-2}z_\rho, \ldots, y_\rho \\ \xi_1^{\rho-1}\zeta_1, \xi_1^{\rho-2}\zeta_1, \ldots, \eta_1 \\ \cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot \\ \xi_\rho^{\rho-1}\zeta_\rho, \xi_\rho^{\rho-2}\zeta_\rho, \ldots, \eta_\rho \end{vmatrix} + \begin{vmatrix} x_1^{\rho}y_1, x_1^{\rho-2}z_1, \ldots, y_1 \\ x_2^{\rho}y_2, x_2^{\rho-2}z_2, \ldots, y_2 \\ \cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot \\ x_\rho^{\rho}y_\rho, x_\rho^{\rho-2}z_\rho, \ldots, y_\rho \\ \xi_1^{\rho}\eta_1, \xi_1^{\rho-2}\zeta_1, \ldots, \eta_1 \\ \cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot \\ \xi_\rho^{\rho}\eta_\rho, \xi_\rho^{\rho-2}\zeta_\rho, \ldots, \eta_\rho \end{vmatrix} = 0.$$

20. As an example of (22) and (23) consider the elliptic functions, *i.e.*, the case in which $\rho=1$; then

$$\mathrm{N}_1 \begin{vmatrix} z & y \\ \zeta & \eta \end{vmatrix} + \begin{vmatrix} xy & y \\ \xi\eta & \eta \end{vmatrix} = 0$$

(dropping suffixes), or

$$\mathrm{N}_1(z^2\eta^2-y^2\zeta^2)=(\xi-x)y\eta(z\eta+y\zeta)$$

that is

$$\mathrm{N}_1\{(x-a_1)(\xi-a_1)-(a_1-a_2)(a_1-a_3)\}=-y\eta(z\eta+y\zeta);$$

and

$$u=\tfrac{1}{2}\int_{a_1}^{x}\frac{dx}{\sqrt{x-a_1.x-a_2.x-a_3}}$$

$$\mathrm{U}=\tfrac{1}{2}\int_{a_1}^{x}\frac{\sqrt{x-a_1}\,dx}{\sqrt{x-a_2.x-a_3}}.$$

Let

$$x=a_1+(a_2-a_1)t^2,\ k^2=\frac{a_2-a_1}{a_3-a_1};$$

then

$$u=\frac{1}{\sqrt{a_3-a_1}}\int_0^t\frac{dt}{\sqrt{1-t^2.1-k^2t^2}},$$

$$\mathrm{U}=\frac{a_2-a_1}{\sqrt{a_3-a_1}}\int_0^t\frac{t^2dt}{\sqrt{1-t^2.1-k^2t^2}}.$$

Let s, c, d denote elliptic functions of $u\sqrt{a_3-a_1}$

$\mathrm{S}, \mathrm{C}, \mathrm{D}$,, ,, $v\sqrt{a_3-a_1}$

σ ,, $\mathrm{sn}\{(u+v)\sqrt{a_3-a_1}\}$ *i.e.*, $-\mathrm{sn}w\sqrt{a_3-a_1}$.

Then

$$s^2=t^2=\frac{x-a_1}{a_2-a_1},$$

and therefore

$$x-a_1=\quad(a_2-a_1)s^2$$
$$x-a_2=-(a_2-a_1)c^2$$
$$x-a_3=-(a_3-a_1)d^2$$

and so

$$N_1=\frac{(a_2-a_1)^2\sqrt{a_3-a_1}\{scdS^2+SCDs^2\}}{(a_2-a_1)(a_3-a_1)(1-k^2s^2S^2)}$$
$$=\frac{a_2-a_1}{\sqrt{a_3-a_1}}sS\sigma.$$

With the ordinary notation for the second elliptic integral we have

$$E(u\sqrt{a_3-a_1})=u\sqrt{a_3-a_1}-\frac{U}{\sqrt{a_3-a_1}}$$

and since

$$u+v+w=0$$

this gives

$$-\frac{1}{\sqrt{a_3-a_1}}(U+V+W)=E(u\sqrt{a_3-a_1})+E(v\sqrt{a_3-a_1})+E(w\sqrt{a_3-a_1}$$
$$=k^2sS\sigma$$
$$=\frac{N_1}{\sqrt{a_3-a_1}}$$

that is

$$U+V+W=-N_1$$

agreeing with the case when $\rho=1$ of (22).

21. The evaluation of N_1 in terms of the functions can be obtained in the general case as follows.

Since $x_1, x_2, \ldots, x_\rho, \xi_1, \xi_2, \ldots, \xi_\rho, p_1, p_2, \ldots, p_\rho$ are the roots of

$$M^2y^2-N^2z^2=0$$

we have

$$M^2y^2-N^2z^2\equiv(x-x_1)(x-x_2)\ldots(x-p_\rho).$$

In this write $x=a_m$, where $m\geqq1\leqq\rho$; then

$$-N^2Q(a_m)\equiv(a_m-x_1)(a_m-x_2)\ldots(a_m-p_\rho)$$

that is

$$l_m\{N(a_m)\}^2=l_m{}^3al_m{}^2(u)al_m{}^2(v)al_m{}^2(u+v)$$

and therefore

$$N_1a_m{}^{\rho-1}+N_2a_m{}^{\rho-2}+\ldots+N_\rho=\pm l_mal_m(u)al_m(v)al_m(u+v).$$

Hence

$$\sum_{m=1}^{m=\rho}\frac{l_m}{P'(a_m)}al_m(u)al_m(v)al_m(u+v)=\pm\sum_{m=1}^{m=\rho}\frac{N_1a_m{}^{\rho-1}+\ldots+N_\rho}{P'(a_m)}.$$

But $a_1, a_2, \ldots a_\rho$ are the ρ roots of

$$\mathrm{P}(z)=0$$

and therefore as before

$$\sum_{m=1}^{m=\rho}\frac{a_m^{\rho-1}}{\mathrm{P}'(a_m)}=1 \text{ and } \sum_{m=1}^{m=\rho}\frac{a_m^{\rho-s}}{\mathrm{P}'(a_m)}=0(s>1\leqq\rho)$$

and so

$$\sum_{m=1}^{m=\rho}\left\{\frac{l_m}{\mathrm{P}'(a_m)}al_m(u)al_m(v)al_m(u+v)\right\}=\pm\mathrm{N}_1$$

and therefore

$$\begin{aligned}\mathrm{U}+\mathrm{V}+\mathrm{W}&=-\mathrm{N}_1\\ &=\mp\sum_{m=1}^{m=\rho}\left\{\frac{l_m}{\mathrm{P}'(a_m)}al_m(u)al_m(v)al_m(u+v)\right\}.\end{aligned}$$

On the expansion of each side in terms of the u's and v's as is done below, it is at once seen that the lower sign is the correct one; and therefore

$$\mathrm{U}+\mathrm{V}+\mathrm{W}=\sum_{m=1}^{m=\rho}\left\{\frac{l_m}{\mathrm{P}'(a_m)}al_m(u)al_m(v)al_m(u+v)\right\}.$$

This may be called the addition theorem for the integral-function; by putting $\rho=1$ and referring to the example worked out in the last section, it is at once seen to be the addition theorem for elliptic integrals of the second order.

22. In the expansion of the two sides in terms of u's and v's the first term is sufficient to indicate the correct sign in the above; but it is not uninteresting to see the agreement for terms of a higher order, and the expansion is carried on as far as the order seven in the magnitudes u.

Proceeding therefore to form the expansion of U in terms of the u's, write, with WEIERSTRASS,

$$\frac{\mathrm{P}'(a_r)}{l_r}(a_r-x_r)=s_r^2 \quad \ldots\ldots\ldots\ldots \quad (24)$$

so that

$$-\frac{\mathrm{P}'(a_r)}{l_r}dx_r=2s_r ds_r$$

and

$$\frac{dx_r}{x_r-a_r}=2\frac{ds_r}{s_r}.$$

Let

$$\mathfrak{a}_{r,m}=\frac{l_r}{\mathrm{P}'(a_r)}\,\frac{1}{a_r-a_m} \quad \ldots\ldots\ldots\ldots \quad (25)$$

so that

$$\frac{l_m}{\mathrm{P}'(a_m)}\mathfrak{a}_{r,m}=-\frac{l_r}{\mathrm{P}'(a_r)}\mathfrak{a}_{m,r};$$

then

$$x_r-a_m=(a_r-a_m)(1-\mathfrak{a}_{r,m}s_r^2).$$

Substituting we have

$$P(x_m)=-\frac{l_m s_m^2}{P'(a_m)}P'(a_m)+\left\{\frac{l_m s_m^2}{P'(a_m)}\right\}^2\frac{P''(a_m)}{2!}-\left\{\frac{l_m s_m^2}{P'(a_m)}\right\}^3\frac{P'''(a_m)}{3!}+\ldots$$

$$Q(x_m)=-l_m-\frac{l_m s_m^2}{P'(a_m)}Q'(a_m)+\left\{\frac{l_m s_m^2}{P'(a_m)}\right\}^2\frac{Q''(a_m)}{2!}-\ldots$$

and therefore

$$\sqrt{\frac{P(x_m)}{Q(x_m)}}=s_m\sqrt{\frac{1-\dfrac{l_m s_m^2}{\{P'(a_m)\}^2}\dfrac{P''(a_m)}{2!}+\ldots}{1+\dfrac{s_m^2}{P'(a_m)}Q'(a_m)-\ldots}}$$

$$=s_m(1-A_m s_m^2-B_m s_m^4)\text{ say, } \quad \ldots\ldots \quad (26)$$

correct to the fifth order. Moreover

$$dU=\tfrac{1}{2}\sum_{m=1}^{m=\rho}\sqrt{\frac{P(x_m)}{Q(x_m)}}dx_m$$

$$=-\sum_{m=1}^{m=\rho}\frac{l_m}{P'(a_m)}s_m^2 ds_m(1-A_m s_m^2-B_m s_m^4)$$

or

$$U=-\sum_{m=1}^{m=\rho}\frac{l_m}{P'(a_m)}\left[\frac{s_m^3}{3}-\frac{A_m}{5}s_m^5-\frac{B_m}{7}s_m^7\right] \quad \ldots\ldots \quad (27)$$

correct to the seventh order. Further

$$2du_m=\sqrt{\frac{P(x_1)}{Q(x_1)}}\frac{dx_1}{x_1-a_m}+\ldots+\sqrt{\frac{P(x_m)}{Q(x_m)}}\frac{dx_m}{x_m-a_m}+\ldots$$

and

$$\frac{dx_r}{x_r-a_m}=-2\mathrm{a}_{r,m}s_r ds_r(1+\mathrm{a}_{r,m}s_r^2+\mathrm{a}_{r,m}^2 s_r^4)$$

which with the help of (26) gives

$$du_m=ds_m(1-A_m s_m^2-B_m s_m^4)-\sum_{r=1}^{r=\rho}{}'\,\mathrm{a}_{r,m}s_r^2 ds_r(1+\mathrm{a}_{r,m}s_r^2+\mathrm{a}_{r,m}^2 s_r^4)(1-A_r s_r^2-B_r s_r^4)$$

where Σ' denotes that r may receive all values between 1 and ρ except m. Thus

$$u_m=s_m-\frac{A_m}{3}s_m^3-\frac{B_m}{5}s_m^5-\sum_{r=1}^{r=\rho}{}'\,\mathrm{a}_{r,m}\left\{\frac{s_r^3}{3}+\frac{\mathrm{a}_{r,m}-A_r}{5}s_r^5\right\} \quad \ldots\ldots \quad (28)$$

correct to the fifth order; and as s_m is of the order u this expansion will be sufficient for the expansion of U in (27) accurately to the seventh order. The equation (28) holds for $m=1, 2, \ldots, \rho$.

Inverting it in order to obtain s in terms of u's we find that to the fifth order

$$
\begin{aligned}
s_m = u_m & \\
& + \frac{A_m}{3} u_m^3 + \overset{r=\rho}{\underset{r=1}{\Sigma^m}} a_{r,m} \frac{u_r^3}{3} \\
& + \frac{5A_m^2+3B_m}{15} u_m^5 + \frac{A_m}{3} u_m^2 \overset{r=\rho}{\underset{r=1}{\Sigma^m}} a_{r,m} u_r^3 + \overset{r=\rho}{\underset{r=1}{\Sigma^m}} \frac{a_{r,m}(3a_{r,m}+2A_r)}{15} u_r^5 + \tfrac{1}{3} \overset{r=\rho}{\underset{r=1}{\Sigma^m}} a_{r,m} u_r^2 \overset{s=\rho}{\underset{s=1}{\Sigma^r}} a_{s,r} u_s^3 \quad (29)
\end{aligned}
$$

where $\overset{r=\rho}{\underset{r=1}{\Sigma^m}}$ implies summation for all values of r from 1 to ρ except m, and $\overset{s=\rho}{\underset{s=1}{\Sigma^r}}$ for all values of s from 1 to ρ except r. Substituting this in (27) we obtain

$$
\begin{aligned}
-U = \sum_{m=1}^{m=\rho} \frac{l_m}{P'(a_m)} \Bigg[& \frac{u_m^3}{3} + \frac{2A_m}{15} u_m^5 + u_m^2 \overset{r=\rho}{\underset{r=1}{\Sigma^m}} a_{r,m} \frac{u_r^3}{3} + \left(\frac{2B_m}{35} + \frac{A_m^2}{9} \right) u_m^7 \\
& + \frac{2A_m}{3} u_m^4 \overset{r=\rho}{\underset{r=1}{\Sigma^m}} a_{r,m} \frac{u_r^3}{3} + \frac{u_m^2}{15} \overset{r=\rho}{\underset{r=1}{\Sigma^m}} a_{r,m}(3a_{r,n}+2A_r) u_r^5 \\
& + \frac{u_m^2}{3} \overset{r=\rho}{\underset{r=1}{\Sigma^m}} a_{r,m} u_r^2 \overset{s=\rho}{\underset{s-1}{\Sigma^r}} a_{s,r} u_s^3 + \frac{u_m}{9} \overset{r=\rho}{\underset{r=1}{\Sigma^m}} \overset{s=\rho}{\underset{s=1}{\Sigma^m}} a_{r,m} a_{s,m} u_r^3 u_s^3 \Bigg] \quad . \quad . \quad . \quad (30)
\end{aligned}
$$

correct to the seventh order. In the last term inside the bracket r and s may take the same value; the double summation is in fact

$$
\left[\overset{r=\rho}{\underset{r=1}{\Sigma^m}} a_{r,m} u_r^3 \right]^2 .
$$

Again

$$
\begin{aligned}
al_m^2(u) & = \frac{1}{l_m} \phi(a_m) \\
& = s_m^2(1-a_{1,m}s_1^2)(1-a_{2,m}s_2^2) \ldots (1-a_{\rho,m}s_\rho^2)
\end{aligned}
$$

(the term involving s_m^2 not occurring in the brackets)

$$
= s_m^2 \left[1 - \overset{r=\rho}{\underset{r=1}{\Sigma^m}} a_{r,m} s_r^2 + \overset{r,t=\rho}{\underset{r,t=1}{\Sigma}} a_{r,m} a_{t,m} s_r^2 s_t^2 \right]
$$

where $\overset{r,t=\rho}{\underset{r,t=1}{\Sigma^m}}$ implies summation for all values 1, 2, . . . , ρ of r and t except m, and r and t must not have the same values. Extracting the square root we find

$$
al_m(u) = s_m \left[1 - \tfrac{1}{2} \overset{r=\rho}{\underset{r=1}{\Sigma^m}} a_{r,m} s_r^2 - \tfrac{1}{8} \overset{r=\rho}{\underset{r=1}{\Sigma^m}} a_{r,m}^2 s_r^4 + \tfrac{1}{4} \overset{r,t=\rho}{\underset{r,t=1}{\Sigma^m}} a_{r,m} a_{t,m} s_r^2 s_t^2 \right] \quad . \quad . \quad . \quad (31).
$$

Let σ_m refer to $al_m(v)$, S_m to $al_m(u+v)$, so that to the first order

$$al_m(v) = \sigma_m$$

$$al_m(u+v) = \mathrm{S}_m$$

and regard σ_m and S_m as being of the same order as s_m. Then accurately to the seventh order

$$al_m(u)al_m(v)al_m(u+v)$$

$$= s_m\sigma_m\mathrm{S}_m\Big[1 - \tfrac{1}{2}\mathop{\Sigma^{m}}\limits_{r=1}^{r=\rho} \mathrm{a}_{r,m}(s_r^2+\sigma_r^2+\mathrm{S}_r^2) - \tfrac{1}{8}\mathop{\Sigma^{m}}\limits_{r=1}^{r=\rho} \mathrm{a}_{r,m}{}^2(s_r^4+\sigma_r^4+\mathrm{S}_r^4)$$

$$+ \tfrac{1}{4}\mathop{\Sigma^{m}}\limits_{r,t=1}^{r,t=\rho} \mathrm{a}_{r,m}\mathrm{a}_{t,m}(s_r^2s_t^2+\sigma_r^2\sigma_t^2+\mathrm{S}_r^2\mathrm{S}_t^2) + \tfrac{1}{4}\mathop{\Sigma^{m}}\limits_{r=1}^{r=\rho}\mathop{\Sigma^{m}}\limits_{t=1}^{t=\rho} \mathrm{a}_{r,m}\mathrm{a}_{t,m}(s_r^2\sigma_t^2+\sigma_r^2\mathrm{S}_t^2+\mathrm{S}_r^2s_t^2)\Big] \quad (32)$$

where the summation in the last term in (32) is exactly as in the last term in (30). To express this in terms of u and v we must substitute the value of s in terms of u's as given by (29) and for σ and S respectively corresponding values of v and $u+v$. Let these values be inserted, both sides multiplied by $\dfrac{l_m}{\mathrm{P}'(a_m)}$ and the summation taken for the values $m=1$ to $m=\rho$ and compare this expression, which is

$$\sum_{m=1}^{m=\rho} \frac{l_m}{\mathrm{P}'(a_m)} al_m(u)al_m(v)al_m(u+v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{B}),$$

with the value of U+V+W.

Firstly, they agree in the third order of quantities; for

$$-\frac{u_m^3+v_m^3+w_m^3}{3} = u_mv_m(u_m+v_m)$$

since

$$u_m+v_m+w_m = 0.$$

Secondly, consider in each the terms of the order five. That in U+V+W which has $\dfrac{l_m}{\mathrm{P}'(a_m)}\mathrm{A}_m$ for its coefficient is

$$\tfrac{2}{15}[(u_m+v_m)^5-u_m^5-v_m^5]$$

$$= \tfrac{2}{3}u_mv_m(u_m^3+2u_m^2v_m+2u_mv_m^2+v_m^3)$$

$$= \tfrac{2}{3}u_mv_m(u_m+v_m)(u_m^2+v_m^2+u_mv_m)$$

while in (B) it is

$$\tfrac{1}{3}u_mv_m(u_m+v_m)\{u_m^2+v_m^2+(u_m+v_m)^2\}$$

and these are obviously equal.

The term in U+V+W which has

$$\tfrac{1}{3}\frac{l_m}{\mathrm{P}'(a_m)}\,\frac{l_r}{\mathrm{P}'(a_r)}\,\frac{1}{a_r-a_m},\ i.e.\ \tfrac{1}{3}\frac{l_m}{\mathrm{P}'(a_m)}\mathrm{a}_{r,m} \text{ or } -\tfrac{1}{3}\frac{l_r}{\mathrm{P}'(a_r)}\mathrm{a}_{m,r},$$

for coefficient is

$$(u_m+v_m)^2(u_r+v_r)^3-(u_r+v_r)^2(u_m+v_m)^3-u_m^2u_r^3+u_r^2u_m^3-v_m^2v_r^3+v_r^2v_m^3$$

while in (B) it is

$$\begin{aligned}&(u_m+v_m)v_mu_r^3-(u_r+v_r)v_ru_m^3-\tfrac{3}{2}u_mv_m(u_m+v_m)\{u_r^2+v_r^2+(u_r+v_r)^2\}\\&+(u_m+v_m)u_mv_r^3-(u_r+v_r)u_rv_m^3+\tfrac{3}{2}u_rv_r(u_r+v_r)\{u_m^2+v_m^2+(u_m+v_m)^2\}\\&+u_mv_m(u_r+v_r)^3-u_rv_r(u_m+v_m)^3.\end{aligned}$$

Adding the latter up in columns, it is

$$\begin{aligned}&u_r^3[(u_m+v_m)^2-u_m^2]+v_r^3[(u_m+v_m)^2-v_m^2]+3u_mv_mu_rv_r(u_r+v_r)\ \text{for first}\\&-u_m^3[u_r+v_r)^2-u_r^2]-v_m^3[(u_r+v_r)^2-v_r^2]-3u_mv_mu_rv_r(u_m+v_m)\ \text{for second}\\&-3u_mv_m(u_m+v_m)(u_r+v_r)^2+3u_mv_mu_rv_r(u_m+v_m)+3u_rv_r(u_r+v_r)(u_m+v_m)^2\\&\qquad\qquad-3u_rv_ru_mv_m(u_r+v_r)\ \text{for third}\\&=-u_m^2u_r^3-v_m^2v_r^3+u_m^3u_r^2+v_m^3v_r^2+(u_m+v_m)^2(u_r+v_r)^3-(u_r+v_r)^2(u_m+v_m)^3\end{aligned}$$

and therefore, to the order five, (B) and U+V+W are equal.

Thirdly, consider in the order seven the term in U+V+W which has

$$\frac{l_m}{\mathrm{P}'(a_m)}\left(\frac{2\mathrm{B}_m}{35}+\frac{\mathrm{A}_m^2}{9}\right)$$

for coefficient; it is

$$\begin{aligned}&(u_m+v_m)^7-u_m^7-v_m^7\\&=7u_mv_m(u_m+v_m)[u_m^4+v_m^4+3u_m^2v_m^2+2u_mv_m(u_m^2+v_m^2)]\end{aligned}$$

while in B the term of order seven which is free from all the a's and is multiplied by $\frac{l_m}{\mathrm{P}'(a_m)}$ is

$$\begin{aligned}&\frac{l_m}{\mathrm{P}'(a_m)}[u_mv_m(u_m+v_m)\{u_m^4+v_m^4+(u_m+v_m)^4\}]\frac{5\mathrm{A}_m^2+3\mathrm{B}_m}{15}\\&+\frac{l_m}{\mathrm{P}'(a_m)}\frac{\mathrm{A}_m^2}{9}[u_mv_m(u_m+v_m)\{v_m^2(u_m+v_m)^2+u_m^2(u_m+v_m)^2+u_m^2v_m^2\}]\\&=\frac{l_m}{\mathrm{P}'(a_m)}\left(\frac{2\mathrm{B}_m}{5}+\frac{7\mathrm{A}_m^2}{9}\right)u_mv_m(u_m+v_m)[u_m^4+v_m^4+3u_m^2v_m^2+2u_mv_m(u_m^2+v_m^2)]\end{aligned}$$

and again these terms are equal.

I have verified the exact agreement of the two expressions for one or two others (but not for all, owing to the labour involved) of the terms of the seventh order; and this exact agreement leads us to infer the truth of the equation

$$U+V+W=\sum_{m=1}^{m=\rho}\frac{l_m}{P'(a_m)}al_m(u)al_m(v)al_m(u+v) \quad . \; . \; . \; . \; . \; . \quad (33)$$

a direct proof of which has already been obtained.

23. Combining (22) and (33) we have

$$U+V+W=-N_1=\sum_{s=1}^{s=\rho}\frac{l_s}{P'(a_s)}al_s(u)al_s(v)al_s(u+v).$$

Therefore

$$\sum_{m=1}^{m=\rho}\left[\frac{\delta U}{\delta u_m}\delta u_m+\frac{\delta V}{\delta v_m}\delta v_m+\frac{\delta W}{\delta w_m}\delta w_m\right]=-\sum_{m=1}^{m=\rho}\left[\frac{\delta N_1}{\delta u_m}\delta u_m+\frac{\delta N_1}{\delta v_m}\delta v_m\right].$$

But by (7)

$$\delta u_m+\delta v_m+\delta w_m=0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7')$$

so that substituting for the δw's and remembering that the δu's and δv's are independent we have

$$\left.\begin{aligned}\frac{\delta U}{\delta u_m}-\frac{\delta W}{\delta w_m}&=-\frac{\delta N_1}{\delta u_m}\\ \frac{\delta V}{\delta v_m}-\frac{\delta W}{\delta w_m}&=-\frac{\delta N_1}{\delta v_m}\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (34).$$

By the first of these

$$\frac{\delta^2 U}{\delta u_m\delta u_n}\delta u_n-\frac{\delta^2 W}{\delta w_m\delta w_n}\delta w_n=-\frac{\delta^2 N_1}{\delta u_m\delta u_n}\delta u_n-\frac{\delta^2 N_1}{\delta u_m\delta v_n}\delta v_n$$

and therefore by (7′)

Similarly

$$\left.\begin{aligned}\frac{\delta^2 U}{\delta u_m\delta u_n}+\frac{\delta^2 W}{\delta w_m\delta w_n}&=-\frac{\delta^2 N_1}{\delta u_m\delta u_n}\\ \frac{\delta^2 W}{\delta w_m\delta w_n}&=-\frac{\delta^2 N_1}{\delta u_m\delta v_n}\\ \frac{\delta^2 V}{\delta v_m\delta v_n}+\frac{\delta^2 W}{\delta w_m\delta w_n}&=-\frac{\delta^2 N_1}{\delta v_m\delta v_n}\\ \frac{\delta^2 W}{\delta w_m\delta w_n}&=-\frac{\delta^2 N_1}{\delta v_m\delta u_n}\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (35)$$

from which we see that N_1 satisfies the series of differential equations

$$\frac{\delta^2 N_1}{\delta u_m\delta v_n}=\frac{\delta^2 N_1}{\delta v_m\delta u_n}$$

of which there are $\frac{1}{2}\rho(\rho-1)$ in all.

24. Returning now to (34) and using (14) we have

$$\frac{l_m}{\mathrm{P}'(a_m)}\{al_m^2(u+v)-al_m^2(u)\}=\sum_{s=1}^{s=\rho}\frac{l_s}{\mathrm{P}'(a_s)}al_s(v)\left\{al_s(u)\frac{\partial al_s(u+v)}{\partial u_m}+al_s(u+v)\frac{\partial al_s u}{\partial u_m}\right\} \quad . \ (36)$$

$$\frac{l_m}{\mathrm{P}'(a_m)}\{al_m^2(u+v)-al_m^2(v)\}=\sum_{s=1}^{s=\rho}\frac{l_s}{\mathrm{P}'(a_s)}al_s(u)\left\{al_s(v)\frac{\partial al_s(u+v)}{\partial v_m}+al_s(u+v)\frac{\partial al_s(v)}{\partial v_m}\right\} \quad . \ (37)$$

and from these by subtraction and noticing that

$$\frac{\partial al_s(u+v)}{\partial u_m}=\frac{\partial al_s(u+v)}{\partial v_m}$$

we have

$$\frac{l_m}{\mathrm{P}'(a_m)}\{al_m^2(u)-al_m^2(v)\}=\sum_{s=1}^{s=\rho}\frac{l_s}{\mathrm{P}'(a_s)}al_s(u+v)\left\{al_s(u)\frac{\partial al_s(v)}{\partial v_m}-al_s(v)\frac{\partial al_s(u)}{\partial u_m}\right\} \quad . \quad (38).$$

Now if s be different from m

$$\frac{\partial al_s(u)}{\partial u_m}=\frac{l_m}{\mathrm{P}'(a_m)}al_m(u)al_{s,m}(u)$$

but this no longer holds when s,m are the same since $al_{m,m}$ is not a recognised function. We proceed as follows to obtain $\dfrac{\partial al_m(u)}{\partial u_m}$:—differentiate both sides of (20) with respect to u_m so that

$$al_{\rho+r}(u)\frac{\partial al_{\rho+r}(u)}{\partial u_m}=-\mathop{{\sum}^{m}}\limits_{s=1}^{s=\rho}\left\{\frac{l_s}{a_s-a_{\rho+r}}\,\frac{al_s(u)}{\mathrm{P}'(a_s)}\,\frac{\partial al_s(u)}{\partial u_m}\right\}-\frac{l_m}{a_m-a_{\rho+r}}\,\frac{al_m(u)}{\mathrm{P}'(a_m)}\,\frac{\partial al_m(u)}{\partial u_m}$$

where $\mathop{{\sum}^{m}}\limits_{s=1}^{s=\rho}$ implies that the value $s=m$ is not to be included in the summation. The equation quoted above (holding for all values of s from 1 to $2\rho+1$) when substituted in the last gives, on division by $\dfrac{l_m}{\mathrm{P}'(a_m)}al_m(u)$,

$$al_{m,\rho+r}(u)al_{\rho+r}(u)+\mathop{{\sum}^{m}}\limits_{s=1}^{s=\rho}\left\{\frac{l_s}{(a_s-a_{\rho+r})\mathrm{P}'(a_s)}al_s(u)al_{s,m}(u)\right\}=-\frac{1}{a_m-a_{\rho+r}}\,\frac{\partial al_m(u)}{\partial u_m}$$

and (38) may now be written in the form

$$\begin{aligned}al_m^2(u)-al_m^2(v)=&\mathop{{\sum}^{m}}\limits_{s=1}^{s=\rho}\frac{l_s}{\mathrm{P}'(a_s)}al_s(u+v)\{al_s(u)al_m(v)al_{s,m}(v)-al_s(v)al_m(u)al_{s,m}(u)\}\\&-al_m(u+v)\mathop{{\sum}^{m}}\limits_{s=1}^{s=\rho}\frac{l_s(a_m-a_{\rho+r})}{(a_s-a_{\rho+r})\mathrm{P}'(a_s)}al_s(v)al_{s,m}(v)al_m(u)-al_s(u)al_{s,m}(u)al_m(v)\\&-(a_m-a_{\rho+r})al_m(u+v)\{al_m(u)al_{\rho+r}(v)al_{m,\rho+r}(v)-al_m(v)al_{\rho+r}(u)al_{m,\rho+r}(u)\} \quad (39).\end{aligned}$$

This equation holds for the values 1, 2, . . . , ρ of m and these ρ equations determine the ρ functions $al_m(u+v)$ for values 1, . . . , ρ of m in terms of functions of u and v. Moreover, r is any one of the numbers 1, 2, . . . , $\rho+1$, so that these equations can have a large variety of forms. We may thus consider the functions $al_m(u+v)(m\leqq\rho)$ as known; the $\rho+1$ functions $al_{\rho+r}(u+v)$ are given in terms of them and therefore ultimately in terms of functions of u and v by the equation

$$al^2_{\rho+r}(u+v)=1-\sum_{s=1}^{s=\rho}{}_m\left\{\frac{l_s}{(a_s-a_{\rho+r})\mathrm{P}'(a_s)}al_s^2(u+v)\right\}.$$

Treating (36) in the same manner as (38) it will yield ρ equations involving the double-suffix functions of $u+v$; this system, together with the relations between them (to which reference has already been made), will furnish the complete solution of the addition theorem for these functions.

Abelian functions of order 2.

25. Consider the particular case of the preceding for which $\rho=2$. We now have

$$\left.\begin{aligned}\mathrm{P}(x)&=(x-a_1)(x-a_2)\\ \mathrm{Q}(x)&=(x-a_3)(x-a_4)(x-a_5)\\ \mathrm{R}(x)&=\mathrm{P}(x)\mathrm{Q}(x)\end{aligned}\right\}$$

$$\left.\begin{aligned}u_1&=\tfrac{1}{2}\int_{a_1}^{x_1}\frac{x-a_2}{\sqrt{\mathrm{R}(x)}}dx+\tfrac{1}{2}\int_{a_2}^{x_2}\frac{x-a_2}{\sqrt{\mathrm{R}(x)}}dx\\ u_2&=\tfrac{1}{2}\int_{a_1}^{x_1}\frac{x-a_1}{\sqrt{\mathrm{R}(x)}}dx+\tfrac{1}{2}\int_{a_2}^{x_2}\frac{x-a_1}{\sqrt{\mathrm{R}(x)}}dx\end{aligned}\right\}$$

$$\phi(x)=(x-x_1)(x-x_2).$$

Write

$$\left.\begin{aligned}x_1-a_r&=\mathrm{a}_r\\ x_2-a_r&=\mathrm{b}_r\end{aligned}\right\}(r=1,\,2,\,3,\,4,\,5)\quad .\ .\ .\ .\ .\ .\ .\ .\ .\quad (1).$$

Also

$$\left.\begin{aligned}l_1,\,l_2&=-\mathrm{Q}(a_1),\ -\mathrm{Q}(a_2)\\ l_3,\,l_4,\,l_5&=\mathrm{P}(a_3),\ \mathrm{P}(a_4),\ \mathrm{P}(a_5)\end{aligned}\right\}\text{respectively}\quad .\ .\ .\ .\ .\ .\ .\quad (2).$$

Then

$$l_s al_s^2=\phi(a_s)=\mathrm{a}_s\mathrm{b}_s\quad .\ .\ .\ .\ .\ .\ .\ .\ .\ .\quad (3)$$

for $s=1$, 2, 3, 4, 5; and

$$al_{r,\,s}=\frac{1}{(x_1-x_2)\sqrt{l_rl_s}}\left[\sqrt{\mathrm{a}_r\mathrm{a}_s\mathrm{bbb}}-\sqrt{\mathrm{b}_r\mathrm{b}_s\mathrm{aaa}}\right]\quad .\ .\ .\ .\ .\ .\quad (4)$$

the suffixes being added to the a's and b's under the radical sign so as to have 1, 2, 3, 4, 5 for the complete system under any one root-sign. Then

$$al_t al_{r,s} = \frac{1}{(x_1-x_2)\sqrt{l_r l_s l_t}}\left[\mathrm{b}_t\sqrt{\mathrm{a}_r\mathrm{a}_s\mathrm{a}_t\mathrm{bb}} - \mathrm{a}_t\sqrt{\mathrm{b}_r\mathrm{b}_s\mathrm{b}_t\mathrm{aa}}\right]$$

and therefore

$$(a_r-a_s)al_t al_{r,s} + (a_s-a_t)al_r al_{s,t} + (a_t-a_r)al_s al_{r,t} = 0 \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Again

$$l_r al_r al_{r,s} = \frac{1}{(x_1-x_2)\sqrt{l_s}}\left[\mathrm{a}_r\sqrt{\mathrm{a}_s\mathrm{bbbb}} - \mathrm{b}_r\sqrt{\mathrm{b}_s\mathrm{aaaa}}\right]$$

and therefore

$$(a_p-a_q)l_r al_r al_{r,s} + (a_q-a_r)l_p al_p al_{p,s} + (a_r-a_p)l_q al_q al_{q,s} = 0 \quad . \quad . \quad . \quad . \quad (6)$$

in which p, q, r, s, t may be any of the numbers 1, 2, 3, 4, 5.

26. Writing $\frac{l_s}{\mathrm{P}'(a_s)} = \alpha_s$ $(s=1, 2)$, equation (39) of the last example gives

$$\begin{aligned} al_1^2(u) - al_1^2(v) = {} & \alpha_2 al_2(u+v)\{al_1(v)al_2(u)al_{1,2}(v) - al_1(u)al_2(v)al_{1,2}(u)\} \\ & + al_1(u+v)\Big[(a_1-a_3)\{al_1(v)al_3(u)al_{1,3}u - al_1(u)al_3(v)al_{1,3}(v)\} \\ & \qquad + \alpha_2\frac{a_1-a_3}{a_2-a_3}\{al_1(v)al_2(u)al_{1,2}(u) - al_1(u)al_2(v)al_{1,2}(v)\}\Big] \end{aligned}$$

$$\begin{aligned} al_2^2(u) - al_2^2(v) = {} & \alpha_1 al_1(u+v)\{al_1(u)al_2(v)al_{1,2}(v) - al_1(v)al_2(u)al_{1,2}(u)\} \\ & + al_2(u+v)\Big[(a_2-a_3)\{al_2(v)al_3(u)al_{2,3}(u) - al_2(u)al_3(v)al_{2,3}(v)\} \\ & \qquad + \alpha_1\frac{a_2-a_3}{a_1-a_3}\{al_1(u)al_2(v)al_{1,2}(u) - al_1(v)al_2(u)al_{1,2}(v)\}\Big] \end{aligned}$$

two equations which determine $al_1(u+v)$, $al_2(u+v)$.

Assuming these known we have

$$al_3^2(u+v) = 1 - \left\{\frac{\alpha_1}{a_1-a_3}al_1^2(u+v) + \frac{\alpha_2}{a_2-a_3}al_2^2(u+v)\right\},$$

$$al_4^2(u+v) = 1 - \left\{\frac{\alpha_1}{a_1-a_4}al_1^2(u+v) + \frac{\alpha_2}{a_2-a_4}al_2^2(u+v)\right\},$$

$$al_5^2(u+v) = 1 - \left\{\frac{\alpha_1}{a_1-a_5}al_1^2(u+v) + \frac{\alpha_2}{a_2-a_5}al_2^2(u+v)\right\}.$$

The equation (36) applied to this case is when $m=1$

$$
\begin{aligned}
al_1^2(u+v)-al_1^2(u)=\alpha_2 al_2(v)\{&al_1(u+v)al_2(u)al_{1,2}(u+v)+al_2(u+v)al_1(u)al_{1,2}(u)\}\\
-(a_1-a_3)al_1(v)\{&al_1(u)al_3(u+v)al_{1,3}(u+v)+al_1(u+v)al_3(u)al_{1,3}(u)\}\\
-\alpha_2\frac{a_1-a_3}{a_2-a_3}al_1(v)\{&al_1(u)al_2(u+v)al_{1,2}(u+v)+al_1(u+v)al_2(u)al_{1,2}(u)\}
\end{aligned}
$$

and when $m=2$ it is

$$
\begin{aligned}
al_2^2(u+v)-al_2^2(u)=\alpha_1 al_1(v)\{&al_1(u)al_2(u+v)al_{1,2}(u+v)+al_1(u+v)al_2(u)al_{1,2}(u)\}\\
-(a_2-a_3)al_2(v)\{&al_2(u)al_3(u+v)al_{2,3}(u+v)+al_2(u+v)al_3(u)al_{2,3}(u)\}\\
-\alpha_1\frac{a_2-a_3}{a_1-a_3}al_2(v)\{&al_2(u)al_1(u+v)al_{1,2}(u+v)+al_2(u+v)al_1(u)al_{1,2}(u)\}.
\end{aligned}
$$

A particular case of (5) is

$$(a_1-a_2)al_3al_{1,2}+(a_2-a_3)al_1al_{2,3}+(a_3-a_1)al_2al_{1,3}=0.$$

These three equations will suffice to determine $al_{1,2}(u+v)$, $al_{2,3}(u+v)$, $al_{3,1}(u+v)$; after which the other functions may be successively obtained from the equations

$$
\begin{aligned}
(a_2-a_3)l_4al_4al_{1,4}&=(a_2-a_4)l_3al_3al_{1,3}+(a_4-a_3)l_2al_2al_{1,2} \quad . \quad . \quad . \quad . \quad (6')\\
(a_4-a_2)al_1al_{2,4}&=(a_4-a_1)al_2al_{1,4}+(a_1-a_2)al_4al_{1,2} \quad . \quad . \quad . \quad . \quad . \quad (5')\\
(a_4-a_3)al_2al_{3,4}&=(a_4-a_2)al_3al_{2,4}+(a_2-a_3)al_4al_{2,3} \quad . \quad . \quad . \quad . \quad . \quad (5')\\
(a_2-a_4)l_5al_5al_{1,5}&=(a_2-a_5)l_4al_4al_{1,4}+(a_5-a_4)l_2al_2al_{1,2} \quad . \quad . \quad . \quad . \quad (6')\\
(a_4-a_5)al_1al_{4,5}&=(a_1-a_5)al_4al_{1,5}+(a_4-a_1)al_5al_{1,4} \quad . \quad . \quad . \quad . \quad . \quad (5')\\
(a_3-a_5)al_4al_{3,5}&=(a_3-a_4)al_5al_{3,4}+(a_4-a_5)al_3al_{4,5} \quad . \quad . \quad . \quad . \quad . \quad (5')\\
(a_2-a_5)al_3al_{2,5}&=(a_2-a_3)al_5al_{2,3}+(a_3-a_5)al_2al_{3,5} \quad . \quad . \quad . \quad . \quad . \quad (5')
\end{aligned}
$$

the figure at the end of each line denoting from which of the equations (5) and (6) the particular line has been derived.

This case has been added and all the necessary equations have been written down as a justification of the statement made at the end of § 24.